1. [40 points: 8 points per part]
   Consider the matrix
   \[
   \mathbf{A} = \begin{bmatrix}
   1 & 3 \\
   2 & 6
   \end{bmatrix}.
   \]
   You may draw the plots requested below in MATLAB, or carefully by hand.
   If you use MATLAB, please use the `axis equal` command, which forces the “x” and “y” axes to use the same scaling. This feature should reveal a critical property of the plots.

   (a) Determine all vectors \( \mathbf{x} \in \mathbb{R}^2 \) that solve \( \mathbf{A}\mathbf{x} = \mathbf{0} \).
   Make a plot of \( \mathbb{R}^2 \), indicating those \( \mathbf{x} = [x_1, x_2]^T \) that solve \( \mathbf{A}\mathbf{x} = \mathbf{0} \) as points on the graph.
   (To denote such a vector \( \mathbf{x} \), draw a dot at the \((x_1, x_2)\) point on your plot.)

   (b) Determine all vectors \( \mathbf{b} \in \mathbb{R}^2 \) for which \( \mathbf{A}^T\mathbf{x} = \mathbf{b} \) has some (perhaps not unique) solution \( \mathbf{x} \).
   Show these vectors \( \mathbf{b} = [b_1, b_2]^T \) on your plot from part (a).
   (Be sure to distinguish the \( \mathbf{x} \) in part (a) from the \( \mathbf{b} \) in part (b), e.g., by using different colors.)

   (c) Repeat part (a), but now finding those \( \mathbf{x} \) that solve \( \mathbf{A}^T\mathbf{x} = \mathbf{0} \).
   Create a new plot that shows these vectors \( \mathbf{x} = [x_1, x_2]^T \).

   (d) Repeat part (b), but now finding those \( \mathbf{b} \) for which \( \mathbf{A}\mathbf{x} = \mathbf{b} \) has some (perhaps not unique) solution \( \mathbf{x} \). Show these \( \mathbf{b} = [b_1, b_2]^T \) on your plot from part (c).

   (e) Find all solutions \( \mathbf{x} \) to \( \mathbf{A}\mathbf{x} = \mathbf{b} \) for \( \mathbf{b} = [2, 4]^T \).
   Show these \( \mathbf{x} = [x_1, x_2]^T \) on your plot from part (a).
   (The original problem statement asked you to put this on your plot from part (c). If you have already completed the problem, that is acceptable; however, the plot is more meaningful in contrast to the graphs you made in parts (a) and (b) above.)

2. [60 points: 7.5 points per part, per structure]
Consider the two structures shown in the diagrams on the previous page. (The straight lines connecting the masses are struts, which behave like the springs in the course notes.) Label the structures in the same way we labeled them in class, and apply horizontal and vertical forces to each mass. These forces displace each mass; say that mass $m_k$ moves $x_{2k-1}$ units down, and $x_{2k}$ units to the right. (In the second structure, the fourth and fifth struts meet the third strut at 45° angles; for a discussion of how to handle oblique springs/struts, consult Section 3.5 of the course notes.)

Complete the following questions for each of the two structures.

(a) Work out (linearized) formulas for the elongation of each strut. (For example, in each case you should have $e_1 = -x_1$.) Organize your equations as $e = Ax$, specifying the entries in $A$.

(b) Complete the steps in our four-step modeling procedure to arrive at the equation $(A^TKA)x = f$. Compute the matrix $A^TKA$ (which will involve the $k_j$ variables).

(c) Characterize all vectors $x$ for which $(A^TKA)x = 0$. To what physical motion do these displacements correspond? (For the simple three-spring truss in the notes, these $x$ correspond to shifting the two masses horizontally by the same amount, with no vertical displacement.)

(d) Describe what struts you should add to make each structure stable, i.e., so that in the resulting model, $A^TKAx = 0$ implies that $x = 0$. (You need not work out formulas, but you should specify how many struts you need, and where they should be placed. There is not a unique answer.)

3. [postponed until Problem Set 3]

Consider these structures, a truss bridge and a pathetic model of the Eiffel Tower. In each plot, o denotes a node (i.e., a mass) that can move in the horizontal and vertical directions; a solid black line denotes a strut (i.e., a spring). Note that the tower has four horizontal struts; the top two might be a bit tough to see in the diagram.

You should answer this problem without writing down the entries in the matrix $A$ for these structures, so do not fret about the odd angles between the struts.

As we will see during lectures, $R(A)$ denotes the column space (or range) of $A$, while $N(A)$ denotes the null space of $A$:

\[
R(A) = \{Ax : x \in \mathbb{R}^n\} = \text{the set of all weighted sums } Ax \text{ of the columns of } A;
\]

\[
N(A) = \{x \in \mathbb{R}^n : Ax = 0\} = \text{the set of all } x \text{ for which } Ax = 0.
\]
(a) For both the bridge and tower, count the numbers of nodes and struts.

(b) As described in class, we can collectively model the horizontal and vertical displacement $x$ of the nodes subject to a load $f$ by solving the equation $(A^T KA)x = f$. The first step of this process computes elongations, $e = Ax$ for $A \in \mathbb{R}^{m \times n}$.

For both the bridge and tower, specify the dimensions $m$ and $n$ of $A$.

(c) In class we will observe that $\text{rank}(A)$ equals the number of pivot columns (and pivot rows) in the reduced form of $A$, and hence equals the dimension of the subspaces $\mathbb{R}(A)$ and $\mathbb{R}(A^T)$.

For both the bridge and tower, what is the largest possible value for the dimension of $\mathbb{R}(A)$? (Use only the matrix dimensions: do not try to write down the entries of $A$.)

(d) Based on your answer to part (c), what is the smallest possible dimension of the subspace $N(A)$ for each structure?

(e) Show that if $x \in N(A)$, then $x \in N(A^T KA)$.

(This means that $N(A) \subseteq N(A^T KA)$, so $N(A)$ cannot be smaller than $N(A^T KA)$.)

(f) We say that a structure is unstable if $N(A^T KA)$ contains nonzero vectors.

Based on the previous parts of this problem, one structure above is guaranteed to be unstable regardless of the stiffness values in $K$. Which structure is unstable, and why?

(g) By assessing the dimensions computed above, what is the minimum number of struts that you must add to the unstable structure so it can potentially be stable?

(h) By assessing the dimensions computed above, what is the minimum number of struts that you must remove from the stable structure to ensure it will be unstable?