1. [60 points: 10 points per part]

Recall the strategy for population modeling described in the first lecture. Suppose the population is now divided into seven age brackets (0–9 years old; 10–19; 20–29; 30–39; 40–49; 50–59; 60+). Associated with each age bracket is a fertility parameter \( f_1, f_2, \ldots, f_7 \) that describes the average number of female children born to a woman in the specified age bracket over ten years’ time, and a survivability parameter \( s_1, s_2, \ldots, s_7 \) giving the probability that a member of the age bracket survives for a decade. Given the initial population in each bracket \( p_{(0)}^1, p_{(0)}^2, \ldots, p_{(0)}^7 \), the population after one ten-year “generation” \( p_{(1)}^1, p_{(1)}^2, \ldots, p_{(1)}^7 \) is given by the seven equations:

\[
\begin{align*}
  p_{(1)}^1 &= f_1 p_{(0)}^1 + f_2 p_{(0)}^2 + f_3 p_{(0)}^3 + f_4 p_{(0)}^4 + f_5 p_{(0)}^5 + f_6 p_{(0)}^6 + f_7 p_{(0)}^7, \\
  p_{(1)}^2 &= s_1 p_{(0)}^1, \\
  p_{(1)}^3 &= s_2 p_{(0)}^2, \\
  p_{(1)}^4 &= s_3 p_{(0)}^3, \\
  p_{(1)}^5 &= s_4 p_{(0)}^4, \\
  p_{(1)}^6 &= s_5 p_{(0)}^5, \\
  p_{(1)}^7 &= s_6 p_{(0)}^6 + s_7 p_{(0)}^7.
\end{align*}
\]

(a) We view these seven coupled equations as a single matrix equation of the form \( \mathbf{p}^{(1)} = \mathbf{A} \mathbf{p}^{(0)} \). Write out the matrix \( \mathbf{A} \) (specify all its entries).

(b) Roughly estimate fertility parameters and survivability parameters for the United States. You may just make (reasonable!) guesses for these numbers for full credit, but if you want to get accurate numbers, you can track down much data at [www.cdc.gov/nchs](http://www.cdc.gov/nchs) and [www.census.gov/people](http://www.census.gov/people) (Remember that we are considering fertility and survivability rates for a ten year period.)

(c) Edit `population.m` (or roll your own code) to illustrate the population growth (or decay) that arises from your the parameters you picked in part (b). By computing \( \mathbf{A}^k \mathbf{p}^{(0)} \) for \( k = 0, 1, 2, \ldots, 50 \), produce a plot of population versus time for generations 0 to 50. (Start by allocating an equal number of members to each age bracket at generation 0.)

(d) We wish to study the growth rate of your population, versus generation. To do so, let

\[
\tau_k = p_{(k)}^1 + p_{(k)}^2 + p_{(k)}^3 + p_{(k)}^4 + p_{(k)}^5 + p_{(k)}^6 + p_{(k)}^7
\]

denote the total population of generation \( k \). Produce a plot showing \( \tau_k / \tau_{k-1} \) on the vertical axis, and \( k = 1, \ldots, 50 \) on the horizontal axis.

(e) Now significantly change the initial distribution of the population; say, put most of the people in the first age bin. Repeat part (d), and use MATLAB’s `hold on` command to superimpose your new plot of the growth rate on the earlier plot. Do they tend to the same limit?

(f) Change your parameters in part (c) to produce a population that balances the fertility and survivability parameters so that the total population neither grows nor decays much. Show a plot as in part (c), and list the fertility and survivability parameters you used.

(You could do this trivially by setting the fertility rates to zero and the survivability rates to one. Please present a more interesting solution!)

please see the next page...
2. [40 points: 20 points for (a); 10 points each for (b) and (c); 5 point bonus]
Consider the following circuit with six resistors.

(a) Work through the first three steps of our circuit modeling procedure:
   (1) compute potential drops, $e = v - Ax$;
   (2) compute the current in each resistor, $y = Ke$;
   (3) apply Kirchhoff’s current law, $0 = A^*y$.

   At each step, write out the individual scalar equations (e.g., $e_1 = v_0 - x_1$), and then the matrix–vector form that collects those scalar equations together. In particular, specify the entries of $x$, $y$, $v$, $e$, and $A$.

(b) Now assume $R_1 = R_2 = R_3 = R_4 = R_5 = R_6 = 1$. Work out the entries of the matrix $S = A^*KA$.

(c) Under the conditions of part (b), use Gaussian elimination (by hand, not MATLAB) to solve the system $(A^*KA)x = A^*Kv$ for the unknown potential vector

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}.$$

   (Since the vector $v$ contains the voltage variable $v_0$, your answer $x$ should also contain the variable $v_0$, as in equation (2.12) in the course notes. Here is a quick check on your answer: notice that the structure of the circuit and the fact that all resistances are equal ensures that $x_2 = x_3$. Does your answer agree?)

(d) [optional: 5 point bonus]
   Work out (by hand, not MATLAB) the LU factorization $S = LU$ of the matrix in part (b), where $L$ is a lower triangular matrix with ones on the diagonal, and $U$ is an upper triangular matrix.