Chapter 3
Simple Structures at Equilibrium

Simple mechanical structures may seem a far cry from the circuits discussed in the last lecture, but the underlying mathematical models are strikingly similar. Here we illustrate how this different physical scenario gives rise to the same $A^T K A x = b$ equation we studied earlier. This observation shows the great merit of developing a mathematical understanding of general systems having this form: in mastering the underlying theory, we develop tools for handling a diversity of applications.

These notes draw heavily in spirit, details, and examples from the texts of Gilbert Strang\textsuperscript{1} and Steve Cox\textsuperscript{2}.

3.1 A springy column

Consider the arrangement of four springs in a vertical column shown in Figure 3.1, with three masses separating the springs. The springs are fixed at the top and bottom of this arrangement, and when forces are applied to these masses, the springs will compress or extend. (For now we do not allow the springs to move left or right out of this vertical arrangement.)

Our goal is to determine how the forces $f_1$, $f_2$, and $f_3$, applied to the masses $m_1$, $m_2$, and $m_3$, affect the displacements $x_1$, $x_2$, and $x_3$. The solution will depend on the material properties of the springs (i.e., the spring constants $k_1$, $k_2$, and $k_3$): we expect stiff springs will allow smaller deformations than more flexible springs. In any case, we presume that our springs behave according to Hooke’s Law; when dealing with real springs, this will not generally be the case if the forces are too small or too great — with the extreme case being the fracture of the spring under an excessive load. (Moreover, while we speak of “springs,” we might instead envision a “truss,” perhaps a steel girder, a timber beam, or a concrete pier that we assume to behave in a roughly Hookean fashion.)

\textsuperscript{1} Gilbert Strang. Introduction to Applied Mathematics. Wellesley-Cambridge Press, Wellesley, MA, 1986
\textsuperscript{2} Steven J. Cox. Matrix Analysis in Situ. Rice University, 2013
The procedure for determining the displacements $x_1$, $x_2$, and $x_3$ will closely resemble our methodology for modeling circuits. Again, we break the process into four steps. Let us agree to measure positive quantities in the down direction; e.g., if the top spring gets longer under the applied load, the displacement $x_1$ of mass $m_1$ will be positive.

**STEP 1 Compute the extension of each spring.**

We first measure the elongation of the four different springs. As seen in Figure 3.1, the loads will cause some springs to stretch while others compress; a positive “elongation” means the spring is stretched, while a negative value indicates compression. The first elongation is easy to compute: it is simply $x_1$, the amount the first mass has descends under the load. We thus set

$$e_1 = x_1.$$  

The amount the second spring stretches equals the amount by which the drop of mass $m_2$ exceeds the drop of mass $m_1$. (In the cartoon in Figure 3.1, the top two masses have dropped by the same amount, so the second spring has zero elongation.) Thus, the extension of the second spring is

$$e_2 = x_2 - x_1.$$  

The third spring stretches similarly:

$$e_3 = x_3 - x_2.$$  

Finally, the last spring get shorter by the amount that mass $m_3$ descends, so

$$e_4 = -x_3.$$  

As usual, we arrange these four equations in matrix-vector form:

$$\begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix},$$ (3.1)

which we write as

$$e = Ax.$$ (3.2)

**STEP 2 Apply Hooke’s Law.**

Next we seek to relate the elongation of spring $j$ to the restoring force $y_j$ that the spring exerts. Hooke’s Law does the trick: the force is proportional to the elongation, with the proportionality given by the spring constant:

$$y_j = k_j e_j, \quad j = 1, \ldots, 4.$$  

The springs will change length, but the sum of the changes must be zero, since the top and bottom are fixed.
In matrix-vector form,

\[
\begin{bmatrix}
y_1 \\
y_2 \\
y_3 \\
y_4 \\
\end{bmatrix} =
\begin{bmatrix}
k_1 & 0 & 0 & 0 \\
0 & k_2 & 0 & 0 \\
0 & 0 & k_3 & 0 \\
0 & 0 & 0 & k_4 \\
\end{bmatrix}
\begin{bmatrix}
e_1 \\
e_2 \\
e_3 \\
e_4 \\
\end{bmatrix},
\]

which we write as

\[y = Ke.\] (3.3)

**STEP 3 Balance forces at each mass.**

We aim to figure out how much the applied forces \(f_1, f_2,\) and \(f_3\) cause the masses to descend at equilibrium. The key step is to balance these known forces acting on each mass against the restoring force of each spring. Since the system is at rest (static), the forces balance at each mass. Getting the signs of the restoring forces correct can be tricky. At mass \(m_1,\) the force exerted by the top spring acts to restore to the spring to its original length, hence it pulls \(m_1 up,\) which, by our convention, is the negative direction. Meanwhile, as the second spring seeks to be restored to its rest length, it tugs mass 1 downward, the positive direction. Hence, the applied force \(f_1\) balances the restoring force \(-y_1 + y_2:\)

\[f_1 - y_1 + y_2 = 0.\]

The same argument applied to masses \(m_2\) and \(m_3\) to give

\[f_2 - y_2 + y_3 = 0\]
\[f_3 - y_3 + y_4 = 0.\]

We rearrange to get the matrix-vector form,

\[
\begin{bmatrix}
1 & -1 & 0 & 0 \\
0 & 1 & -1 & 0 \\
0 & 0 & 1 & -1 \\
\end{bmatrix}
\begin{bmatrix}
y_1 \\
y_2 \\
y_3 \\
y_4 \\
\end{bmatrix} =
\begin{bmatrix}
f_1 \\
f_2 \\
f_3 \\
\end{bmatrix}.
\] (3.4)

The matrix here encodes the connectivity of the spring network, mapping spring forces to masses; in perfect parallel to the circuit model, it is the transpose of the matrix in (3.1) from step 1, which mapped mass displacements to spring extensions: hence we write (3.4) as

\[A^T y = f.\] (3.5)

**STEP 4 Assembly.**

Now we simply put the pieces together to arrive at an equation for the unknown \(x\). Inserting equation (3.3) into (3.5) gives

\[f = A^T y = A^T Ke.\]
Now insert equation (3.2) for \( e \) to give the fundamental relation

\[
A^T K A x = f, \tag{3.6}
\]

the same equation we arrived at for our circuit model.

**STUDENT EXPERIMENTS**

3.7. Suppose all the springs are identical, \( k_1 = k_2 = k_3 = k_4 = k \), as are the masses, \( m_1 = m_2 = m_3 = m \), and the applied forces come from gravity: \( f_j = mg \). Set up and solve (3.6) for \( x_1 \), \( x_2 \), and \( x_3 \).

3.8. Generalize the configuration in Figure 3.1 to have \( N \) equal masses and \( N + 1 \) identical springs. Fix the total mass in the system, independent of \( N \), and divide it evenly across the masses. How do the displacements \( x_1, \ldots, x_N \) behave as \( N \) gets large? How should the spring constant \( k \) scale with \( N \) so that your solution tends to a clean limit as \( N \to \infty \)?

Ultimately we seek to understand when equations like \( A^T K A x = f \) have a solution, and when that solution is unique. The one-dimensional truss is very clean: you can see that indeed a solution always exists, and it is unique, for the matrix \( A^T K A \) is invertible. Yet we ultimately we will extend this modeling procedure to handle more interesting two-dimensional trusses, and the solvability question gets much more interesting.

### 3.2 Two dimensions and linearization

Circuits do not notice the angles at which we arrange the wires relative to the nodes, but mechanical structures certainly do. We avoided these concerns in the last lecture by forcing all the displacements to occur in the same direction. However, we mostly care about structures in two or three dimensions, where the springs can attach at odd angles. This geometry makes it more difficult to compute the elongation of the springs. To set the stage, consider the tipsy two-dimensional table shown in Figure 3.2. We presume that \( L_1 = L_3 \), so that at rest, the horizontal and vertical springs join at right angles.

Suppose, following Step 1 of the procedure outlined in the last lecture, we want to compute the elongation of the spring on the left, which at rest has length \( L_1 \). When forces are applied, mass \( m_1 \) moves...
$x_1$ units down and $x_2$ units to the right. Use the Pythagorean Theorem to compute the length of the deformed spring,

\[
\text{loaded length of spring 1} = \sqrt{(L_1 - x_1)^2 + x_2^2},
\]

from which we deduce the formula

\[
\text{elongation of spring 1} = \sqrt{(L_1 - x_1)^2 + x_2^2} - L_1. \tag{3.7}
\]

Similar formulas hold for the other two springs:

\[
\text{elongation of spring 2} = \sqrt{((x_1 - x_3)^2 + (L_2 + x_4 - x_2)^2 - L_2} \tag{3.8}
\]

\[
\text{elongation of spring 3} = \sqrt{(L_3 - x_3)^2 + x_4^2} - L_3. \tag{3.9}
\]

Following the pattern of the last lecture, we should now try to write these elongation equations in the matrix-vector form

\[
e = Ax.
\]

However, we run into a fundamental obstacle: equations (3.7)–(3.9) are \textit{nonlinear} functions of $x_1, x_2, x_3,$ and $x_4$; they involve squares, square roots, and products like $x_1 x_3$. There is no way to write these elongations as \textit{linear} functions of the $x_j$ variables, as implied by the equation $e = Ax$.

Despite the exaggerated elongations shown in Figure 3.2, we are generally envisioning deformations that are quite modest, compared to the rest lengths of the springs. In this parameter regime, the elongations are \textit{approximately linear} functions of the deformations. To see this, take a closer look at the elongation of spring 1, which we can rewrite as

\[
\text{elongation of spring 1} = \sqrt{(L_1 - x_1)^2 + x_2^2} - L_1
\]

\[
= \sqrt{L_1^2 \left(1 - \frac{x_1}{L_1}\right)^2 + \left(\frac{x_2}{L_1}\right)^2} - L_1
\]

\[
= L_1 \sqrt{1 - \frac{2x_1}{L_1} + \frac{x_1^2 + x_2^2}{L_1^2}} - L_1. \tag{3.10}
\]
If the rest length $L_1$ is much larger in magnitude than the deformations $x_1$ and $x_2$, then the term $-2x_1/L_1 + (x_1^2 + x_2^2)/L_1^2$ under the radical will be quite small. This calls to mind the Taylor series

$$\sqrt{1 + x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} + \cdots$$

whose leading terms give an excellent approximation when $|x|$ is small. Substituting the square root from (3.10) into the Taylor series, we get

$$\sqrt{1 - \frac{2x_1}{L_1} + \frac{x_1^2 + x_2^2}{L_1^2}} = 1 + \frac{1}{2} \left( -\frac{2x_1}{L_1} + \frac{x_1^2 + x_2^2}{L_1^2} \right) + \frac{1}{8} \left( -\frac{2x_1}{L_1} + \frac{x_1^2 + x_2^2}{L_1^2} \right)^2 + \cdots$$

$$= 1 - \frac{x_1}{L_1} + O\left( \frac{x_1^2 + x_2^2}{L_1^2} \right)$$

$$\approx 1 - \frac{x_1}{L_1}. $$

Now insert this approximation into the elongation formula (3.10):

$$\text{elongation of spring 1} \approx L_1 \left( 1 - \frac{x_1}{L_1} \right) - L_1 = -x_1. \quad (3.11)$$

This approximation seems quite reasonable, as it matches the formula we would have obtained if spring 1 were constrained to only deform in the vertical direction (like the springy column in the last lecture). Next, apply the same approximation strategy to the elongation formulas (3.8) and (3.9) to obtain

$$\text{elongation of spring 2} \approx x_4 - x_2 \quad (3.12)$$

$$\text{elongation of spring 3} \approx -x_3 \quad (3.13)$$

Notice a key property of the approximations (3.11)–(3.13):

| The dominant deformation occurs in the direction of the spring’s orientation. |

That is, the approximate elongation formula is the same thing we would obtain if the spring were constrained to only deform in the direction of its main axis. This general rule makes it easy to approximate the elongation of springs at arbitrary orientations, when we approach structures with more interesting geometry than the one shown in Figure 3.2.

**Student experiments**

3.9. Use geometry to derive the extension formulas (3.8) and (3.9).
3.10. Test the quality of the approximation. Suppose $L_1 = 10$ m, and let $x_1 = x_2$. Produce a plot that compares the true elongation (3.7) of spring 1 to the approximation (3.11) for small displacements starting at $x_1 = x_2 = 0$ m and increasing. (Plot $x_1 = x_2$ on the horizontal axis, and the values of the elongation and its approximation on the vertical axis.) How large can $x_1 = x_2$ be before the approximation (3.11) noticeably loses its accuracy?

3.11. Work out the approximations for the elongations of springs 2 and 3 given in (3.12) and (3.13).

3.3 Four–steps for the linearized model

With the approximate elongations at hand, we can proceed with the four steps of the modeling process described in the last lecture.

**STEP 1** **Compute the (approximate) extension of each spring.**

Since we are after a linear relationship between the displacements and elongations, we use the approximations (3.11)–(3.13):

\[
\begin{align*}
e_1 &= -x_1 \\
e_2 &= x_4 - x_2 \\
e_3 &= -x_3.
\end{align*}
\]

These approximations can be written in the matrix–vector form

\[
\begin{bmatrix}
e_1 \\
e_2 \\
e_3
\end{bmatrix} =
\begin{bmatrix}-1 & 0 & 0 & 0 \\
0 & -1 & 0 & 1 \\
0 & 0 & -1 & 0
\end{bmatrix}
\begin{bmatrix}x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix},
\]

which, of course, we write as

\[
e = Ax.
\]

**STEP 2** **Apply Hooke’s Law.**

Hooke’s Law pays no heed to the way the springs are connected, so there is no need to make a linearizing approximation here. We proceed as before, computing the restoring force in each spring as

\[
y_j = k_j e_j, \quad j = 1, 2, 3,
\]

which takes the matrix form

\[
\begin{bmatrix}y_1 \\
y_2 \\
y_3
\end{bmatrix} =
\begin{bmatrix}k_1 & 0 & 0 \\
0 & k_2 & 0 \\
0 & 0 & k_3
\end{bmatrix}
\begin{bmatrix}e_1 \\
e_2 \\
e_3
\end{bmatrix},
\]
or
\[ y = Ke. \]

**STEP 3  Balance Forces.**

The truss can move within the plane, with forces acting in the vertical and horizontal directions. Hence we must balance forces in two directions at each node. Since (in our linear approximation) the spring elongation occurs in the direction in which the spring is oriented, the force balance step inherits the same assumption. For this simple case, the restoring forces of springs 1 and 3 act in the vertical direction, while spring 2 acts horizontally. So the force balance equations become:

- Mass 1, vertical: \( 0 = f_1 + y_1; \)
- Mass 1, horizontal: \( 0 = f_2 + y_2; \)
- Mass 2, vertical: \( 0 = f_3 + y_3; \)
- Mass 2, horizontal: \( 0 = f_4 - y_2. \)

Collect these four equations as
\[
\begin{bmatrix}
-1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & -1 \\
0 & 1 & 0 \\
\end{bmatrix}
\begin{bmatrix}
y_1 \\
y_2 \\
y_3 \\
y_4 \\
\end{bmatrix} =
\begin{bmatrix}
f_1 \\
f_2 \\
f_3 \\
f_4 \\
\end{bmatrix}.
\]

Yes indeed, the matrix in this last equation once again is \( A^T \), so
\[ A^T y = f. \]

**STEP 4  Assembly.**

This step proceeds just as before:

\[
f = A^T y = A^T Ke = A^T KAx,
\]
resulting in the equation
\[ A^T KAx = f. \] (3.15)

Before proceeding to solve this equation, pause for a moment to consider the matrix dimensions that arise when modeling general planar (2d) trusses. Suppose we have \( m \) masses connected by \( n \) springs. This gives the following dimensions.
These dimensions play a crucial role when it comes to solving for the displacements $x$ corresponding to a known load $f$.

### 3.4 When Gaussian elimination fails

Return now to the specific scenario sketched in Figure 3.2. Work out

$$A^T K A$$

and, with known loads $f_1, f_2, f_3, f_4$, proceed to solve

$$A^T K A x = f,$$

Suppose we try to do so with Gaussian elimination, forming the augmented matrix

$$
\begin{bmatrix}
  k_1 & 0 & 0 & 0 & f_1 \\
  0 & k_2 & 0 & -k_2 & f_2 \\
  0 & 0 & k_3 & 0 & f_3 \\
  0 & -k_2 & 0 & k_2 & f_4
\end{bmatrix}.
$$

Given the large number of zeros in this matrix, elimination looks to be an easy task. First we target the (4,2) entry, which can be eliminated by replacing row 4 by the sum of rows 2 and 4:

$$
\begin{bmatrix}
  k_1 & 0 & 0 & 0 & f_1 \\
  0 & k_2 & 0 & -k_2 & f_2 \\
  0 & 0 & k_3 & 0 & f_3 \\
  0 & 0 & 0 & 0 & f_4 + f_2
\end{bmatrix}.
$$

What just happened? We have an upper triangular matrix on the left, but a strange one. Perhaps it helps to write out the equations:

$$k_1 x_1 = f_1$$

(3.17)
\[ k_2 x_2 - k_2 x_4 = f_2 \quad (3.18) \]
\[ k_3 x_3 = f_3 \quad (3.19) \]
\[ 0 = f_2 + f_4. \quad (3.20) \]

The last of these equations imposes a **consistency condition**. Remember that we seek \( x_1, x_2, x_3, \) and \( x_4 \) that satisfy a **static equilibrium**: if the condition \( f_2 + f_4 = 0 \) is violated, then the horizontal load is imbalanced \( (f_2 \neq -f_4) \), and the structure will not stand: no static equilibrium exists. You knew this instinctively from Figure 3.2, and now you see it confirmed by the linear algebra.

But what if the forces do balance, \( f_2 + f_4 = 0 \)? Then equations \((3.17)\) and \((3.19)\) immediately give

\[ x_1 = f_1/k_1, \quad x_3 = f_3/k_3. \]

Regarding \( x_2 \) and \( x_4 \), we only know

\[ x_2 = x_4 + f_2/k_2. \]

We call \( x_4 \) a **free variable**: for any value it takes, we can construct a solution to \((3.17)-(3.20)\). Thus, any vector of the form

\[
\begin{bmatrix}
  x_1 \\
  x_2 \\
  x_3 \\
  x_4
\end{bmatrix} =
\begin{bmatrix}
  f_1/k_1 \\
  \gamma + f_2/k_2 \\
  f_3/k_3 \\
  \gamma
\end{bmatrix}, \quad f_2 = -f_4 \quad (3.21)
\]

solves \( A^T K A x = f \). That is, there are **infinitely many** static configurations of the structure for any single consistent load.

One particular choice of consistent forces illuminates the key problem. Consider the trivial load

\[ f_1 = f_2 = f_3 = f_4 = 0, \]

in which case the solution space \((3.21)\) becomes

\[
\begin{bmatrix}
  x_1 \\
  x_2 \\
  x_3 \\
  x_4
\end{bmatrix} =
\begin{bmatrix}
  0 \\
  \gamma \\
  0 \\
  \gamma
\end{bmatrix}, \quad f = 0 \quad (3.22)
\]

for any value of \( \gamma \). Think about the physical significance of this solution. When no load is applied, the structure is free to displace in any of the solutions \((3.22)\), which correspond to both masses shifting to the right (or left) by the same amount. A structure that permits such unforced shifts cannot stand: it is **unstable**, and the linear algebra reveals the nature of this instability.

**Student Experiments**

...aside perhaps from the pile of debris that occurs when the structure collapses!
3.12. How would you change the tipsy table in Figure 3.2 to stabilize it? How would your modification affect the linear algebra leading to the equation $A^T K A x = f$?

Now, in the simple arrangement of Figure 3.2 the trouble is easy to diagnose. In structures with hundreds or thousands of struts, instabilities can be much more difficult to eyeball. We will thus develop tools for diagnosing these instabilities, and, later, for understanding near-instabilities too. Before doing so, we should fix that instability.

3.5 *Trusses with oblique supports*

Our analysis of the truss in Figure 3.2 was simplified by the fact that the springs meet at right angles. More interesting structures inevitably present more complicated geometry. How do we resolve springs at oblique angles?

Suppose mass $m_j$ is connected to mass $m_k$ by spring $\ell$, forming an angle $\theta$ measured clockwise from the horizontal, as illustrated in Figure 3.3 Then our approximation rule (that springs are mainly deformed in their direction of rest orientation) gives

\[
elongation \approx \sin(\theta) (\text{net vertical displacement}) + \cos(\theta) (\text{net horizontal displacement})
\]

and so

\[
e_\ell = \sin(\theta) (x_{2k-1} - x_{2j-1}) + \cos(\theta) (x_{2k} - x_{2j}). \quad (3.23)
\]

Try checking this formula by applying it to the horizontal and vertical springs that make up the truss in Figure 3.2.

Now put equation (3.23) into action in a more interesting scenario, by introducing a diagonal brace to support the truss in Figure 3.2. Suppose for simplicity that the three springs in the original table have equal length, $L_1 = L_2 = L_3$, with a new diagonal spring connecting $m_1$ to the floor, forming an angle of $\pi/4$ with the second spring, as shown in Figure 3.4. We quickly recapitulate the steps of our modeling procedure. The first three springs elongate as before:

\[
e_1 = -x_1 \\
e_2 = x_4 - x_2 \\
e_3 = -x_3
\]

The fourth spring is obviously the interesting one. Appealing to (3.23) with $\theta = \pi/4$ gives

\[
e_4 = \sin(\pi/4)(0 - x_1) + \cos(\pi/4)(0 - x_2)
\]
\[-\frac{\sqrt{2}}{2} (x_1 + x_2).\]

Now we set up the elongation equations in the form \( e = Ax \):

\[
\begin{bmatrix}
  e_1 \\
  e_2 \\
  e_3 \\
  e_4
\end{bmatrix} =
\begin{bmatrix}
  -1 & 0 & 0 & 0 \\
  0 & -1 & 0 & 1 \\
  0 & 0 & -1 & 0 \\
  -\sqrt{2}/2 & -\sqrt{2}/2 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
  x_1 \\
  x_2 \\
  x_3 \\
  x_4
\end{bmatrix}.
\]

Step 2 of our modeling procedure proceeds as expected:

\[
\begin{bmatrix}
  y_1 \\
  y_2 \\
  y_3 \\
  y_4
\end{bmatrix} =
\begin{bmatrix}
  k_1 & 0 & 0 & 0 \\
  0 & k_2 & 0 & 0 \\
  0 & 0 & k_3 & 0 \\
  0 & 0 & 0 & k_4
\end{bmatrix}
\begin{bmatrix}
  e_1 \\
  e_2 \\
  e_3 \\
  e_4
\end{bmatrix}.
\]

Step 3 requires that the restoring force of each spring be resolved into its horizontal and vertical components. For the scenario sketched in Figure 3.3, spring \( \ell \) connects to masses \( m_j \) and \( m_k \), and so exerts forces on both bodies. Those forces must be resolved into horizontal and vertical components.

<table>
<thead>
<tr>
<th>mass</th>
<th>direction</th>
<th>contribution</th>
</tr>
</thead>
<tbody>
<tr>
<td>( m_j )</td>
<td>vertical</td>
<td>( \sin(\theta_{\ell}) y_{\ell} )</td>
</tr>
<tr>
<td>( m_j )</td>
<td>horizontal</td>
<td>( \cos(\theta_{\ell}) y_{\ell} )</td>
</tr>
<tr>
<td>( m_k )</td>
<td>vertical</td>
<td>( -\sin(\theta_{\ell}) y_{\ell} )</td>
</tr>
<tr>
<td>( m_k )</td>
<td>horizontal</td>
<td>( -\cos(\theta_{\ell}) y_{\ell} )</td>
</tr>
</tbody>
</table>

Thus, in our scenario we still have four force balance equations, but with new terms for the diagonal springs:

**MASS 1, VERTICAL:** \( 0 = f_1 + y_1 + (\sqrt{2}/2) y_4 \);

**MASS 1, HORIZONTAL:** \( 0 = f_2 + y_2 + (\sqrt{2}/2) y_4 \);

**MASS 2, VERTICAL:** \( 0 = f_3 + y_3 \);

**MASS 2, HORIZONTAL:** \( 0 = f_4 - y_2 \).

Thus we arrive at the balance equation

\[
\begin{bmatrix}
  -1 & 0 & -\sqrt{2}/2 \\
  0 & -1 & -\sqrt{2}/2 \\
  0 & 0 & -1 \\
  0 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
  y_1 \\
  y_2 \\
  y_3 \\
  y_4
\end{bmatrix} =
\begin{bmatrix}
  f_1 \\
  f_2 \\
  f_3 \\
  f_4
\end{bmatrix}.
\]

Indeed, this is our usual equation

\[ A^T y = f, \]

leading once more to

\[ A^T K A x = f. \]

In light of the instability we diagnosed for the original tipsy table, we now ask the critical question:

Here we have taken \( j = 1 \) in (3.23), and since the spring anchored into the floor at the far end, the displacements \( x_{2k-1} \) and \( x_{2k} \) are both zero.
Will the brace stabilize the structure, i.e., will $A^T K A x = 0$ only have the trivial solution $x = 0$?

Work out

$$A^T K A = \begin{bmatrix} k_1 + k_4/2 & k_4/2 & 0 & 0 \\
 k_4/2 & k_2 + k_4/2 & 0 & -k_2 \\
 0 & 0 & k_3 & 0 \\
 0 & -k_2 & 0 & k_2 \end{bmatrix}.$$

To simplify the arithmetic, suppose $k_1 = k_2 = k_3 = k_4 = 1$. Then to explore solutions to $A^T K A x = f$, set up the augmented matrix

$$\begin{bmatrix} 3/2 & 1/2 & 0 & 0 & f_1 \\
 1/2 & 3/2 & 0 & -1 & f_2 \\
 0 & 0 & 1 & 0 & f_3 \\
 0 & -1 & 0 & 1 & f_4 \end{bmatrix}.$$

Eliminate the $(2,1)$ entry:

$$\begin{bmatrix} 3/2 & 1/2 & 0 & 0 & f_1 \\
 0 & 4/3 & 0 & -1 & f_2 - f_1/3 \\
 0 & 0 & 1 & 0 & f_3 \\
 0 & -1 & 0 & 1 & f_4 \end{bmatrix}.$$

Now eliminate the $(4,2)$ entry:

$$\begin{bmatrix} 3/2 & 1/2 & 0 & 0 & f_1 \\
 0 & 4/3 & 0 & -1 & f_2 - f_1/3 \\
 0 & 0 & 1 & 0 & f_3 \\
 0 & 0 & 0 & 1/4 & f_4 + 3f_3/4 - f_1/4 \end{bmatrix},$$

reducing the matrix on the left to upper-triangular form. Unlike the previous augmented form (3.16), all then entries on the main diagonal of this upper triangular matrix are nonzero. This means that for any choice of $f$, we can find a unique corresponding $x$ vector. In this case, the augmented triangular form implies

$$\frac{3}{2} x_1 + \frac{1}{2} x_2 = f_1$$
$$\frac{3}{4} x_2 - x_4 = f_2 - \frac{1}{4} f_1$$
$$x_3 = f_3$$
$$\frac{1}{4} x_4 = f_4 - \frac{3}{4} f_2 - \frac{1}{4} f_1,$$

giving, for each $f$, the unique solution

$$x = \begin{bmatrix} x_1 \\
 x_2 \\
 x_3 \\
 x_4 \end{bmatrix} = \begin{bmatrix} f_1 - f_2 - f_4 \\
 -f_1 + 3f_2 + 3f_4 \\
 f_3 \\
 -f_1 + 3f_2 + 4f_4 \end{bmatrix}.$$
The only way for $\mathbf{A}^T \mathbf{K} \mathbf{x} = \mathbf{f}$ to equal zero is for $\mathbf{f} = \mathbf{0}$, so we conclude that the table with the diagonal brace is stable. The same result holds any choice of (nonzero) spring constants, $k_1, k_2, k_3,$ and $k_4$. With the diagonal spring added, the instability has been removed, and the structure sits in its static equilibrium.

In the next lecture, we shall delve more deeply into the scenarios where $\mathbf{A}^T \mathbf{K} \mathbf{x} = \mathbf{f}$ has a unique solution.