CHAPTER 3: INTRODUCTION TO SETS

Section 3.1: Operations on Sets

Terms and Notation

Undefined Terms: Every definition uses terms other than the one being defined. If we insist that each of these terms also be defined, those definitions would introduce other terms whose definitions would, in turn, introduce even more terms. Eventually, we would find that our definitons have circled. In mathematics, we wish to avoid circularity, so certain basic terms are declared to be **undefined**. The concepts of point, plane, and set are examples of terms that we declare as undefined.

Sets: Since "set" is, by choice, an undefined term, we give only an intuitive statement of what a set is. Thus, we understand a **set** to be a collection of objects. We insist, however, that a set, say A, satisfy the following two requirements:

(1) The elements of A must come from some well understood **universal set**. For example, the set of all sets is not an acceptable universal set.
(2) For each element x in the universal set, the sentence "x is an element of A" must be a statement; that is, the sentence must be true or false. For example, if n is a particular integer, the sentence "n is a large integer" is not a statement, so we will not permit the set A consisting of all large integers.

Elements of a Set: Let A be a set with universal set U. If x is an element in U we write $x \in A$ to mean that x **is an element of** A, and we write $x \notin A$ to denote that x **is not an element of** A.

The empty set, denoted by the symbol $\emptyset$, is the set that contains no elements.

Cardinality: If A is a set, the number of elements in A is called the **cardinality** of A and will be denoted by $|A|$.

Set Notation: We use set braces $\{\ldots\}$ to enclose the elements of a set. For small finite sets, we can actually list the elements of the set. For example, if $A = \{\, 1, 2, 3 \,\}$ then $1 \in A$ but $4 \notin A$. More commonly, we describe a set using **set-builder notation** wherein a set is defined using the form

$$A = \{\, x \in U \mid P(x) \,\}.$$

In this notation, the vertical line, "|", is read "such that" and $P(x)$ is a statement. Thus, A is the truth set of the statement $P(x)$.

Example 1: If

$$A = \{ n \in \mathbf{N} \,|\, n \text{ is prime and } n \leq 10 \}$$

then $A = \{ 2, 3, 5, 7 \}$. On the other hand, the set

$$B = \{ n \in \mathbf{N} \,|\, n \text{ is prime and } n \geq 10 \}$$

is infinite, so we cannot list its elements.

Exercise 1: (cf. the review for Exercise 2.4.7) Let M_2 denote the universal set of all 2×2 matrices with real number entries. Set

$$S = \{ A \in M_2 \,|\, \det(A) = 1 \}.$$

(a) Exhibit three matrices in M_2 that are elements of S.
(b) Argue that $|S|$ is infinite.
(c) Prove that for all 2×2 matrices A and B, if $A \in S$ and $B \in S$, then $AB \in S$.

We conclude this subsection by noting that in listing the elements of a set, **order and repetition are not relevant**. For example

$$\{1, 2, 3\} = \{3, 2, 1\} = \{1, 2, 2, 3, 3, 3\}.$$

An Aside: Russell's Paradox

Suppose there exists one underlying universal set, U that contains everything. Then, among other things, every set A is an element in U. Now for all sets A with which we are familiar, $A \notin A$. Let us then form the set B defined by

$$B = \{ A \in U \,|\, A \notin A \}.$$

Since U contains everything, including all sets, we must have $B \in U$. Now either $B \in B$ or $B \notin B$. But if $B \in B$ then, by definition of B, $B \notin B$, a contradiction. On the other hand, if $B \notin B$ then, by the defintion of B, we get $B \in B$, again a contradiction.
This paradox leads to the conclusion that there is no single great universal set; that is, nothing contains everything.

Operations on Sets:

Definitions: Let A and B be sets in the universal set U.

The **union** of A and B, denoted $A \bigcup B$ and read "A union B", is the set

$$A \bigcup B = \{ x \in U \,|\, x \in A \text{ or } x \in B \}.$$

The **intersection** of A and B, denoted $A \bigcap B$ and read "A intersect B", is the set

$$A \bigcap B = \{ x \in U \,|\, x \in A \text{ and } x \in B \}.$$

The **complement of** B **relative to** A, denoted $A - B$ and read "A minus B", is the set

$$A - B = \{ x \in U \,|\, x \in A \text{ and } x \notin B \}.$$

The **complement of a set** A, denoted by A' is the set $U - A$; that is

$$A' = \{ x \in U \,|\, x \notin A \}.$$

Example 2: Let $A = \{1, 2, 3, 4\}$ and $B = \{3, 4, 5, 6, 7\}$. Then $A \bigcup B = \{1, 2, 3, 4, 5, 6, 7\}$, $A \bigcap B = \{3, 4\}$, and $A - B = \{1, 2\}$.

Definition: If A and B are sets, the **cartesian product of** A **and** B, denoted $A \times B$ and read "A cross B", is the set

$$A \times B = \{ (a, b) \,|\, a \in A \text{ and } b \in B \}.$$

More generally, if A_1, $A_2, \ldots, A_n$ are sets then

$$A_1 \times A_2 \times \cdots \times A_n = \{ (a_1, a_2, \ldots, a_n) \,|\, a_1 \in A_1, a_2 \in A_2, \ldots, a_n \in A_n \}.$$

Note that an element $(a_1, a_2, \ldots, a_n) \in A_1 \times A_2 \times \cdots \times A_n$ is an **ordered n-tuple**; that is, the order in which the entries are listed matters. For instance, in $\mathbf{Z} \times \mathbf{Z}$, $(1, 2) \neq (2, 1)$.

Example 3: Let $A = \{1, 2\}$ and $B = \{a, b, c\}$. Then

$A \times B = \{ (1, a), (1, b), (1, c), (2, a), (2, b), (2, c) \}$,

$B \times A = \{ (a, 1), (a, 2), (b, 1), (b, 2), (c, 1), (c, 2) \}$, and

$A \times A = \{ (1, 1), (1, 2), (2, 1), (2, 2) \}$.

Theorem: Let A and B be finite sets. Then $A \times B$ is also a finite set and $|\, A \times B \,| = |A|\,|B|$.

Proof: If $(a, b) \in A \times B$ then there are $|A|$ choices for a and $|B|$ choices for b. Thus, there are $|A|\,|B|$ choices for the element (a, b); that is, $|\, A \times B \,| = |A|\,|B|$.

SECTION 3.1: EXERCISES

Set-Builder Notation

3.1.1. (a) Let $A = \{\, 2^n + 1 \mid n \text{ is a prime integer} \,\}$. List the smallest 4 elements of A.

(b) Let $B = \{\, 2^n + 1 \mid n \in \mathbf{Z}^+ \text{ and } 2^n + 1 \text{ is a prime integer} \,\}$. List the smallest 4 elements of B.

(c) Let $C = \{\, n \in \mathbf{Z}^+ \mid 2^n + 1 \text{ is a prime integer} \,\}$. List the smallest 4 elements of C.

3.1.2. Set $A = \{\, n \in \mathbf{Z} \mid 6n^2 - 7n + 2 = 0 \,\}$. List the elements of A.

Notation for 3.1.3 - 3.1.5: In Exercises 3.1.3 - 3.1.5 let $A = \mathbf{R} - \{-2/3\}$ (so A is the set of all real numbers except $-2/3$) and define $f : A \to \mathbf{R}$ by $f(x) = (2x - 1)/(3x + 2)$. The **range of** f is the set

$$R(f) = \{\, y \in \mathbf{R} \mid \text{there exists } x \in A \text{ such that } y = f(x) \,\}.$$

3.1.3. The object of this exercise is to prove that $1 \in R(f)$.

(a) Using the definition of $R(f)$ given above, complete the statement:

$1 \in R(f)$ provided ....

(b) Give a preliminary construction of the proof. In particular, derive the existence of an appropriate $x \in A$.

(c) Give a proof that $1 \in R(f)$.
Check your proof against guidelines 1 - 7 of Section 2.5.

3.1.4 The object of this exercise is to prove, by contradiction, that $\frac{2}{3} \notin R(f)$.

(a) Using the definition of $R(f)$ given above, complete the statement:

$\frac{2}{3} \notin R(f)$ provided ....

(b) Give a proof, by contradiction, that $\frac{2}{3} \notin R(f)$.
Check your proof against guidelines 1 - 7 of Section 2.5.

3.1.5. Let $B = \mathbf{R} - \{\, 2/3 \,\}$. The object of this exercise is to prove that for every $y \in B$, $y \in R(f)$; That is, we want to prove:

$$(\forall y \in B)(\exists x \in A)\, f(x) = y.$$

(a) Construct a preliminary proof. In particular, derive the existence of x, expressed in terms of y.

(b) Prove the given statement. Check your proof against guidelines 1 - 7 of Section 2.5.
[WARNING: Your proof must include an argument that $x \in A$; that is, $x \in \mathbf{R}$ and $x \neq -\frac{2}{3}$. The latter is easily proved by contradiction.]

3.1.6. Let $A = \{ (x, y) \in \mathbf{R} \times \mathbf{R} \mid x + 3y = 0 \}$.

(a) List 3 specific elements of A. Argue that A is an infinite set.

(b) Prove that for all (a, b), $(c, d) \in \mathbf{R} \times \mathbf{R}$, if (a, b), $(c, d) \in A$, then $(a + c, b + d) \in A$. Check your proof against guidelines 1 - 7 of Section 2.5.

3.1.7. Let $A = \{ f : \mathbf{R} \to \mathbf{R} \mid f''$ exists and for all $x \in \mathbf{R}$, $f''(x) + f(x) = 0 \}$.

(a) Verify that if $h(x) = \sin x$ and $k(x) = \cos x$ then $h, k \in A$.

(b) Prove that for all $f, g \in A$ and for all $a, b \in \mathbf{R}$, if $q(x) = af(x) + bg(x)$ then $q \in A$. Check your proof against guidelines 1 - 7 of Section 2.5.

Operations on Sets

3.1.8. In this problem take the universal set to be $U = \{ 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 \}$, let $A = \{ 1, 4, 6 \}$, $B = \{ 1, 3, 4, 7, 8 \}$, and $C = \{ 1, 4, 9 \}$. Exhibit each of the following sets:

(a) $A \bigcup B$ (b) $A \bigcap B$ (c) $B - A$ (d) $A - B$ (e) B'

(f) $(A \bigcup B)'$ (g) $A \times B$.

3.1.9. Let a, b, c, and d be real numbers such that $a < b < c < d$.

(a) Express the set $[a, b] \bigcup [c, d]$ as the difference of two intervals. [No proof required.]

(b) Express the set $[a, c] \bigcap [b, d]$ as the difference of two intervals. [No proof required.]

SECTION 3.2: RELATIONS BETWEEN SETS

Subsets: Definition:

Definition 1: For sets A and B in the universal set U we say that A **is a subset of** B, written $A \subseteq B$, provided for every $x \in U$, if $x \in A$ then $x \in B$.

NOTE: If $x \in A$ we do NOT write $x \subseteq A$. It would be correct, however, to write $\{x\} \subseteq A$. Likewise, it would be incorrect to write $\{x\} \in A$.

Exercise 1: Let $A = \{1, \{1\}, \{1,2\}\}$. Determine $|A|$ and label each of the following as true of false.

(a) $1 \in A$. (b) $1 \subseteq A$. (c) $\{1\} \subseteq A$. (d) $\{1\} \in A$.
(e) $\{\{1\}\} \subseteq A$. (f) $2 \in A$. (g) $\{2\} \subseteq A$. (h) $\{1,2\} \in A$.
(i) $\{1,2\} \subseteq A$. (j) $\{\{1,2\}\} \subseteq A$. (k) $\emptyset \in A$. (l) $\emptyset \subseteq A$.

Theorem 1: For every set A, $\emptyset \subseteq A$.

Comment: In symbols, $\emptyset \subseteq A$ means $(\forall x \in U_x)\,(x \in \emptyset \to x \in A)$. Since the hypothesis, $x \in \emptyset$, is always false, the implication, $x \in \emptyset \to x \in A$, is always true.

Exercise 2: Label each of the following as true or false.

(a) $\emptyset = \{\emptyset\}$ (b) $\emptyset \in \{\emptyset\}$ (c) $\emptyset \subseteq \{\emptyset\}$ (d) $|\{\emptyset\}| = 1$.

Definition 2: Let A be a set. The **power set** of A, denoted $P(A)$ is the set whose elements are the subsets of A.

Example 1: If $A = \{1,2,3\}$ then

$$P(A) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}.$$

Theorem 2: Let A be a set with $|A| = n$. Then $|P(A)| = 2^n$.

Exercise 3: Use Definition 1 above to complete the following definition:

For sets A and B in the universal set U we say that A **is a not subset of** B, written $A \not\subseteq B$, provided ....

Proving that $A \not\subseteq B$:

To Prove: For given sets A and B, $A \not\subseteq B$.

Comment: By the definition (cf Exercise 3), $A \not\subseteq B$ means, in symbolic form,

$$(\exists x \in U)\,(x \in A \,\wedge\, x \notin B).$$

Form of Proof:

- Let $x = a$ where a is a specific element in U that you have derived.
- Verify that $a \in A$; and
- verify that $a \notin B$.

Example 2: Let $A = \{\, x \in \mathbf{R} \,|\, x^2 - 4 > 0 \,\}$ and let $B = \{\, x \in \mathbf{R} \,|\, x^2 - 9 > 0 \,\}$. Prove that $A \not\subseteq B$.

Proof: Let $x = 3$. Then $x^2 - 4 = 3^2 - 4 = 9 - 4 = 5$ and $5 > 0$, so $3 \in A$. But $x^2 - 9 = 3^2 - 9 = 9 - 9 = 0$, so $3 \notin B$. Therefore $A \not\subseteq B$.

Exercise 4: Let $f(x) = x^2 - 3$ and $g(x) = x^2 - 2$. Set
$A = \{\, y \in \mathbf{R} \,|\, \text{there exists } x \in \mathbf{R} \text{ such that } f(x) = y \,\}$ and set
$B = \{\, y \in \mathbf{R} \,|\, \text{there exists } x \in \mathbf{R} \text{ such that } g(x) = y \,\}$.
The object of this exercise is to prove that $A \not\subseteq B$.

Form of the Proof: We must prove:

$$(\exists y \in \mathbf{R})\,(y \in A \,\wedge\, y \notin B)$$

.

Form of Proof:

- Let $y = b$ where b is a specific real number we have derived.
- Verify that $b \in A$.
- Verify that $b \notin B$.

As is often the case, the second and third steps in the form above are proofs within a proof, so we will expand our form to include each of these.

To show $b \in A$ we must prove $(\exists x \in \mathbf{R})\, f(x) = b$.

To show that $b \notin B$ we must show $(\forall x \in \mathbf{R})\, g(x) \neq b$. Proof by contradiction often works well for statements of this sort; that is, statements that something doesn't happen.

We now repeat the form of the proof, with more detail:

Form of Proof:

- Let $y = b$ where b is a specific real number we have derived.
- Verify that $b \in A$.
  - Set $x = a$ where a is a specific real number we have derived.
  - Verify that $f(a) = b$.
- Verify, by contradiction, that $b \notin B$.
  - Assume $b \in B$.
  - Use the definition of B to expand on the previous assumption.
  - Give a logical argument that leads to a contradiction.

Exercise: Follow the form above to write a proof that $A \not\subseteq B$.

Proper Subsets:

Definition 3: For sets A and B in the universal set U we say that A **is a proper subset of** B, written $A \subset B$, provided $A \subseteq B$ and $A \neq B$.

Equivalently, $A \subset B$, provided $A \subseteq B$ and $B \not\subseteq A$. Thus, in symbols, $A \subset B$ means:

$$\Big[(\forall x \in U)\,(x \in A \;\rightarrow\; x \in B)\Big] \;\wedge\; \Big[(\exists y \in U)\,(y \in B \,\wedge\, y \notin A)\Big].$$

To Prove: $A \subset B$

Form of Proof: First, prove that $A \subseteq B$.

- Let x be arbitrary (variable) in U. (If it contributes, expand on what it means to be in U.)
- Suppose $x \in A$. (If it helps, expand on what $x \in A$ means.)
- Now give a logical argument that concludes with $x \in B$.

Now show that $B \not\subseteq A$.

- Let $y = b$, where b is a specific element in U.
- Verify that $b \in B$; and
- Verify that $b \notin A$.

Example 3 (cf. Example 2): Let $A = \{\, x \in \mathbf{R} \,|\, x^2 - 4 > 0 \,\}$ and let $B = \{\, x \in \mathbf{R} \,|\, x^2 - 9 > 0 \,\}$. Prove that $B \subset A$.

Proof: To see that $B \subseteq A$, let $x \in \mathbf{R}$ and suppose $x \in B$. Then $x^2 - 9 > 0$. Now $x^2 - 4 = (x^2 - 9) + 5$. Since both $x^2 - 9$ and 5 are positive, it follows that their sum is positive; that is, $x^2 - 4 > 0$. Therefore, $x \in A$.

To see that $A \not\subseteq B$ (as in Example 2) let $x = 3$. Then $x^2 - 4 = 3^2 - 4 = 9 - 4 = 5$ and $5 > 0$, so $3 \in A$. But $x^2 - 9 = 3^2 - 9 = 9 - 9 = 0$, so $3 \notin B$.

This proves that $B \subset A$.

Exercise 5: Let $M_2(\mathbf{R})$ denote the set of all 2×2 matrices with real number entries, let $O_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, and let $C = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$.

Now set $S = \{ A \in M_2(\mathbf{R}) \mid A = \begin{bmatrix} -3a & a \\ 0 & 0 \end{bmatrix}$ where $a \in \mathbf{R} \}$ and let

$T = \{ B \in M_2(\mathbf{R}) \mid BC = O_2 \}$.

Prove that $S \subset T$.

Set Equality

Definition 4: Let A and B be sets in the universal set U. Then A and B are **equal**, written $A = B$, provided $A \subseteq B$ and $B \subseteq A$.

Comment: To prove that $A = B$ typically requires two proofs:

- $A \subseteq B$; that is, $(\forall x \in U) \Big((x \in A) \to (x \in B) \Big)$; and
- $B \subseteq A$; that is, $(\forall y \in U) \Big((y \in B) \to (y \in A) \Big)$.

To Prove: $A = B$

Form of Proof: First, prove that $A \subseteq B$.

- Let x be arbitrary (variable) in U. (If it contributes, expand on what it means to be in U.)
- Suppose $x \in A$. (If it helps, expand on what $x \in A$ means.)
- Now give a logical argument that concludes with $x \in B$.

Now prove that $B \subseteq A$.

- Let x be arbitrary (variable) in U. (If it contributes, expand on what it means to be in U.)
- Suppose $x \in B$. (If it helps, expand on what $x \in B$ means.)
- Now give a logical argument that concludes with $x \in A$.

Since we have shown that $A \subseteq B$ and $B \subseteq A$, it follows that $A = B$.

Example 4: Let $A = \{ x \in \mathbf{R} \mid -3 < x < 2 \}$ and let $B = \{ x \in \mathbf{R} \mid x^2 + x - 6 < 0 \}$. Prove that $A = B$.

Proof. To see that $A \subseteq B$, let $x \in \mathbf{R}$ and suppose that $x \in A$. Then $-3 < x < 2$. Since $x > -3$, $x + 3$ is positive and since $x < 2$, $x - 2$ is negative. Therefore, $(x + 3)(x - 2)$ is negative; that is, $x^2 + x - 6 = (x + 3)(x - 2) < 0$. This proves that $x \in B$.

To see that $B \subseteq A$, let $x \in \mathbf{R}$ and suppose that $x \in B$. Thus, $x^2 + x - 6 < 0$. Factoring the left side gives $(x+3)(x-2) < 0$. Since the product of $x+3$ and $x-2$ is negative, one of the terms must be negative and the other positive. There are two cases.

Case 1: Suppose that $x + 3 < 0$ and $x - 2 > 0$. This gives $x < -3$ **AND** $x > 2$. This is impossible, so this case cannot occur.

Case 2: Suppose that $x + 3 > 0$ and $x - 2 < 0$. This gives $x > -3$ and $x < 2$; that is, $-3 < x < 2$. Therefore $x \in A$

This proves that $B \subseteq A$ so we conclude that $A = B$.

Exercise 6 (cf. Exercise 5): Let $M_2(\mathbf{R})$ denote the set of all 2×2 matrices with real number entries, let O_2 denote the 2×2 zero matrix, and set $C = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$. Let
$S = \{\, A \in M_2(\mathbf{R}) \mid A = \begin{bmatrix} -3a & a \\ -3b & b \end{bmatrix} \text{ where } a, b \in \mathbf{R} \,\}$ and let
$T = \{\, B \in M_2(\mathbf{R}) \mid BC = O_2 \,\}$.

(a) Do some preliminary proof construction. In particular, for arbitrary $B \in M_2(\mathbf{R})$ assume that $B \in T$ and explore the implications of that assumption.

(b) Prove that $S = T$. Check your proof against guidelines 1 - 7 of Section 2.5.

Set Equality and Abstract Sets:

Exercise 7: Use the definitions given in section 2.1 to complete the following for arbitrary sets A and B.

(a) For $x \in U$, $x \notin A \bigcup B$ provided ....

(b) For $x \in U$, $x \notin A \bigcap B$ provided ....

(c) For $x \in U$, $x \notin A - B$ provided ....

Example 5: Prove or disprove: For all sets A, B, and C, $(A \bigcup B) - C = A \bigcup (B - C)$.

Construction: We must first decide whether the given equality is true of false. The following Venn diagrams will assist toward that end.

$(A \bigcup B) - C$ $\qquad\qquad$ $A \bigcup (B - C)$

Since the Venn diagrams suggest (but do not prove) that the sets are not equal, we will disprove the given statement. Thus we must prove the negation:
$(\exists A, B, C)\left((A \bigcup B) - C \neq A \bigcup (B - C)\right)$. Therefore, we need display specific sets A, B, and C, then verify that $(A \bigcup B) - C \neq A \bigcup (B - C)$ for those sets.

Proof: Let $A = \{1, 2, 3\}$, $B = \{2, 3, 4\}$, and $C = \{1, 4, 6\}$. Then
$(A \bigcup B) - C = \{1, 2, 3, 4\} - \{1, 4, 6\} = \{2, 3\}$, whereas
$A \bigcup (B - C) = \{1, 2, 3\} \bigcup \{2, 3\} = \{1, 2, 3\}$. Therefore, $(A \bigcup B) - C \neq A \bigcup (B - C)$.

Example 6: Prove or disprove: For all sets A, B, and C,
$A - (B \bigcap C) = (A - B) \bigcup (A - C)$.

Construction: Again, we use Venn diagrams to determine whether the sets are equal.

$A - (B \bigcap C)$ $\qquad\qquad$ $(A - B) \bigcup (A - C)$

Since the Venn diagrams suggest (but do not prove) that the sets are equal, we will now prove that they are equal.

Proof: Let A, B, and C be sets. To see that $A - (B \bigcap C) \subseteq (A - B) \bigcup (A - C)$, let $x \in A - (B \bigcap C)$. Then $x \in A$ and $x \notin (B \bigcap C)$. Therefore (cf. Exercise 7(a) above) either $x \notin B$ or $x \notin C$. We consider each possiblity as a separate case.

Case 1: Suppose that $x \notin B$. Then $x \in A$ and $x \notin B$, so $x \in A - B$. It follows that $x \in (A - B) \bigcup (A - C)$.

Case 2: Suppose that $x \notin C$. Then $x \in A$ and $x \notin C$, so $x \in A - C$. It follows that $x \in (A - B) \bigcup (A - C)$.

This proves that $A - (B \bigcap C) \subseteq (A - B) \bigcup (A - C)$.

To see that $(A - B) \bigcup (A - C) \subseteq A - (B \bigcap C)$, suppose $y \in (A - B) \bigcup (A - C)$. Then either $y \in A - B$ or $y \in A - C$. We consider each of these possibilities as a separate case.

Case 1: Suppose $y \in A - B$. Then $y \in A$ and $y \notin B$. Since $y \notin B$, $y \notin B \bigcap C$. Therefore, $y \in A - (B \bigcap C)$.

Case 2: Suppose $y \in A - C$. Then $y \in A$ and $y \notin C$. Since $y \notin C$, $y \notin B \bigcap C$. Therefore, $y \in A - (B \bigcap C)$.

This proves that $(A - B) \bigcup (A - C) \subseteq A - (B \bigcap C)$, so we conclude that the sets are equal.

Basic Set Identities

Theorem 3: Let A, B, and C be sets.

1. **Idempotent Laws**: (a) $A \bigcup A = A$ (b) $A \bigcap A = A$
2. **Commutative Laws**: (a) $A \bigcup B = B \bigcup A$ (b) $A \bigcap B = B \bigcap A$
3. **Associative Laws**: (a) $(A \bigcup B) \bigcup C = A \bigcup (B \bigcup C)$
(b) $(A \bigcap B) \bigcap C = A \bigcap (B \bigcap C)$
4. **Distributive Laws**: (a) $A \bigcap (B \bigcup C) = (A \bigcap B) \bigcup (A \bigcap C)$
(b) $A \bigcup (B \bigcap C) = (A \bigcup B) \bigcap (A \bigcup C)$.
5. **De Morgan's Laws**: (a) $(A \bigcup B)' = A' \bigcap B'$ (b) $(A \bigcap B)' = A' \bigcup B'$
6. **Double Complement Law**: $(A')' = A$
7. **Set Difference**: $A - B = A \bigcap B'$.

Proof of 4(a): We will prove that $A \bigcap (B \bigcup C) = (A \bigcap B) \bigcup (A \bigcap C)$.
To see that $A \bigcap (B \bigcup C) \subseteq (A \bigcap B) \bigcup (A \bigcap C)$, let $x \in A \bigcap (B \bigcup C)$. Then $x \in A$ and $x \in B \bigcup C$. Thus, either $x \in B$ or $x \in C$. We consider each possibility as a separate case:
Case 1: Suppose $x \in B$. Then $x \in A \bigcap B$ so it follows that $x \in (A \bigcap B) \bigcup (A \bigcap C)$.
Case 2: Suppose $x \in C$. Then $x \in A \bigcap C$ so it follows that $x \in (A \bigcap B) \bigcup (A \bigcap C)$.
It now follows that $A \bigcap (B \bigcup C) \subseteq (A \bigcap B) \bigcup (A \bigcap C)$.

To see that $(A \bigcap B) \bigcup (A \bigcap C) \subseteq A \bigcap (B \bigcup C)$, let $y \in (A \bigcap B) \bigcup (A \bigcap C)$. Then either $y \in (A \bigcap B)$ or $y \in (A \bigcap C)$. We consider each of these cases.
Case 1: Suppose $y \in (A \bigcap B)$. Then $y \in A$ and $y \in B$. It follows that $y \in A$ and $y \in B \bigcup C$; that is, $y \in A \bigcap (B \bigcup C)$.
Case 2: Suppose $y \in (A \bigcap C)$. Then $y \in A$ and $y \in C$. It follows that $y \in A$ and $y \in B \bigcup C$; that is, $y \in A \bigcap (B \bigcup C)$.
This proves that $(A \bigcap B) \bigcup (A \bigcap C) \subseteq A \bigcap (B \bigcup C)$ so it follows that $(A \bigcap B) \bigcup (A \bigcap C) = A \bigcap (B \bigcup C)$.

Comment: Note that De Morgan's Laws (Theorem 3(5)) are an immediate consequence of the definitions from Exercise 7.

Example 7: (cf. Example 6) Use the basic set identities given in Theorem 3 to prove that for all sets A, B, and C, $A - (B \bigcap C) = (A - B) \bigcup (A - C)$.

Proof: Let A, B, and C be sets. Then using, in order, Theorem 3 parts 7, 5b, 4a, and 7 we get
$A - (B \bigcap C) = A \bigcap (B \bigcap C)' = A \bigcap (B' \bigcup C') = (A \bigcap B') \bigcup (A \bigcap B') = (A - B) \bigcup (A - C)$.

Exercise 8: Use the basic set identities given in Theorem 3 to prove that for all sets A, B, and C, $(A - B) \bigcup C = (A \bigcup C) - (B - C)$.

SECTION 3.2: EXERCISES

Subsets

3.2.1. Complete the following definition:

For sets A and B, A is not a subset of B, written $A \not\subseteq B$, provided ....

3.2.2. Let $A = \{\, 2^n - 1 \mid n \text{ is a prime integer} \,\}$ and let B be the set of all prime integers. Show that $A \not\subseteq B$ and $B \not\subseteq A$.

3.2.3. Let $A = \{\, x \in \mathbf{R} \mid x^2 - 9 \leq 0 \text{ and } x^2 - 4 > 0 \,\}$ and let $B = \{\, x \in \mathbf{R} \mid 2 < x \leq 3 \,\}$.

(a) Prove that $A \not\subseteq B$.

(b) Using (a), complete a proof that B is a proper subset of A.

3.2.4. Let $A = \{\, (x, y, z) \in \mathbf{R} \times \mathbf{R} \times \mathbf{R} \mid x - y + 5z = 0 \text{ and } 2x - y + 7z = 0 \,\}$ and let $B = \{(u, v, w) \in \mathbf{R} \times \mathbf{R} \times \mathbf{R} \mid (u, v, w) = (-2c, 3c, c) \text{ for some } c \in \mathbf{R} \,\}$. Prove that $A = B$.

3.2.5. **Review**: Recall that if A and B are 2×2 matrices defined by $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} w & x \\ y & z \end{bmatrix}$, then the product of A and B is given by $AB = \begin{bmatrix} aw + by & ax + bz \\ cw + dy & cx + dz \end{bmatrix}$. In general, $AB \neq BA$.

Exercise: Set $P = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$ and let

$$C(P) = \{\, A \in M_2(\mathbf{R}) \mid PA = AP \,\}.$$

Thus $C(P)$ is the set of all matrices that commute with P under matrix multiplication. Also, set

$$S = \{\, B \in M_2(\mathbf{R}) \mid B = \begin{bmatrix} r & s \\ 0 & r + s \end{bmatrix} \text{ for some } r, s \in \mathbf{R} \,\}.$$

Prove that $C(P) = S$.

3.2.6. (a) For arbitrary sets A, B and C, sketch Venn diagrams for each of the sets $(A - B) - C$ and $A - (B - C)$.

(b) Use an elementwise argument to prove that for all sets A, B, and C, $(A - B) - C \subseteq A - (B - C)$.

(c) Prove or disprove: For all sets A, B, and C, $(A - B) - C = A - (B - C)$.

3.2.7. (a) For arbitrary sets A, B and C, sketch Venn diagrams for each of the sets $A - (B \bigcup C)$ and $(A - B) \bigcup (A - C)$.

(b) Prove or disprove: For all sets A, B, and C, $A - (B \bigcup C) = (A - B) \bigcup (A - C)$..

3.2.8. Use an elementwise argument to prove Theorem 3, 4(b); that is, prove for all sets A, B, and C, $A \bigcup (B \bigcap C) = (A \bigcup B) \bigcap (A \bigcup C)$.

3.2.9. Use basic set equalities to prove each of the following for all sets A, B and C.

(a) $A - (B \bigcup C) = (A - B) \bigcap (A - C)$ (b) $(A \bigcup B) - (A \bigcap B) = (A - B) \bigcup (B - A)$.

3.2.10. Prove by contradiction that for all sets A, B, and C, $[(A \bigcap B) - (B \bigcap C)] - (A \bigcap C)' = \emptyset$. [NOTE: If a set is assumed to be nonempty, then we can assert the existence of an element, x, in that set.]

3.2.11. Prove by contrapositive that for all sets A, B, C, and D, if $(A \bigcap B) \subseteq (C \bigcup D)$ then $(A - C) \bigcap (B - D) = \emptyset$. [NOTE: If a set is assumed to be nonempty, then we can assert the existence of an element, x, in that set.]

SECTION 3.3: INDEXED FAMILIES OF SETS

Example 1: For each real number each $r \in (1, \infty)$ define a set A_r by

$$A_r = \{\, x \in \mathbf{R} \,|\, \frac{1}{r} \leq x \leq r \,\}.$$

For instance

$$A_2 = \{\, x \in \mathbf{R} \,|\, \frac{1}{2} \leq x \leq 2 \,\} = [\frac{1}{2}, 2]$$

and

$$A_\pi = \{\, x \in \mathbf{R} \,|\, \frac{1}{\pi} \leq x \leq \pi \,\} = [\frac{1}{\pi}, \pi].$$

The collection of all A_r, where $r \in (1, \infty)$ is an example of an **indexed family of sets** and we denote such a family of sets by

$$\{\, A_r \,\}_{r \in (1,\infty)} \qquad \text{or by} \qquad \{\, A_r \,|\, r \in (1, \infty) \,\}.$$

In this example, the set $(1, \infty)$ is called the **indexing set** for the family. We can think of $(1, \infty)$ as a set of labels for our sets.

NOTES

- In the above family of sets there is no set labeled A_1, there is no "first" set in the family, and given a set A_r in the family, there is no "next" set in the family. The point is, an infinite collection of sets cannot necessarily be labeled simply as A_1, A_2, A_3, etc.
- For each set, A_r, in the above family, the index or label r is associated naturally with the set.
- The indexing set $(1, \infty)$ contains exactly one label for each set in the family.

Example 2: Recall that $\mathbf{R}^2 = \mathbf{R} \times \mathbf{R}$. Geometrically, $\mathbf{R}^2$ denotes the set of points in the plane. For each point $(a, b) \in \mathbf{R}^2$ define a set $P_{(a,b)}$ by

$$P_{(a,b)} = \{\, (x, y) \in \mathbf{R}^2 \,|\, x \geq a \text{ and } y \geq b \,\}.$$

For instance,

$$P_{(1,2)} = \{\, (x, y) \in \mathbf{R}^2 \,|\, x \geq 1 \text{ and } y \geq 2 \,\}.$$

Geometrically, $P_{(1,2)}$ consists of those points in the plane to the right of or on the vertical line $x = 1$ and above or on the horizontal line $y = 2$.

$\{\, P_{(a,b)} \,\}_{(a,b) \in \mathbf{R}^2}$ is an example of an indexed family of sets in which $\mathbf{R}^2$ is the indexing set, or set of labels.

To explore properties of indexed families of sets requires notation for arbitrary indexed families. We will use notation such as $\{ A_\alpha \}_{\alpha\in\Lambda}$ or $\{ B_i \}_{i\in I}$ to denote an abstract indexed family of sets.

Definitions: Let $\{ A_\alpha \}_{\alpha\in\Lambda}$ be an indexed family of sets contained in the universal set U. Then

$$\bigcup_{\alpha\in\Lambda} A_\alpha = \{ x \in U \mid x \in A_\alpha \text{ for some } \alpha \in \Lambda \}.$$

Similarly,

$$\bigcap_{\alpha\in\Lambda} A_\alpha = \{ x \in U \mid x \in A_\alpha \text{ for all } \alpha \in \Lambda \}.$$

Example 3: For each real number $r \in [1, \infty)$ set

$$A_r = \{ x \in \mathbf{R} \mid -\frac{1}{r} \leq x \leq 2 - \frac{1}{r} \}.$$

Thus, in interval notation, $A_r = [-\frac{1}{r}, 2 - \frac{1}{r}]$.

(a) Describe $\bigcup_{r\in[1,\infty)} A_r$.

(b) Prove that the answer given in (a) is correct.

(c) Describe $\bigcap_{r\in[1,\infty)} A_r$.

Solution: To gain some intuition, we first exhibit some examples of A_r. Thus, $A_1 = [-1, 1]$, $A_2 = [-\frac{1}{2}, \frac{3}{2}]$, and $A_3 = [-\frac{1}{3}, \frac{5}{3}]$.

(a) $\bigcup_{r\in[1,\infty)} A_r = [-1, 2)$.

(b) To see that $\bigcup_{r\in[1,\infty)} A_r \subseteq [-1, 2)$, let $x \in \mathbf{R}$ and suppose $x \in \bigcup_{r\in[1,\infty)} A_r$. Then there exists $r \in [1, \infty)$ such that $x \in A_r$. Then $x \in [-\frac{1}{r}, 2 - \frac{1}{r}]$; that is, $-\frac{1}{r} \leq x \leq 2 - \frac{1}{r}$. Therefore $-1 \leq -\frac{1}{r} \leq x \leq 2 - \frac{1}{r} < 2$, so $x \in [-1, 2)$.

To see that $[-1, 2) \subseteq \bigcup_{r\in[1,\infty)} A_r$, let $x \in \mathbf{R}$ and suppose $x \in [-1, 2)$. We will consider two cases.

Case 1: Suppose $x \in [-1, 1]$. Since $A_1 = [-1, 1]$, in this case we have $x \in A_1$, so $x \in \bigcup_{r\in[1,\infty)} A_r$.

Case 2: Suppose $x \in (1, 2)$.

Some Construction: (not included with the proof) We need to find a real number $r \geq 1$ so that $x \in A_r = [-\frac{1}{r}, 2 - \frac{1}{r}]$. Since $x > 1$, let's focus on finding r so that $x \leq 2 - \frac{1}{r}$. To simplify further, let's find r so that $x = 2 - \frac{1}{r}$. Solving for r gives $r = \frac{1}{2-x}$. The proof continues (uninterrupted) as follows.

Set $r = \frac{1}{2-x}$. Since $x \in (1, 2)$, $2 - x < 1$, so $r = \frac{1}{2-x} > 1$; that is, $r \in [1, \infty)$. Further, $2 - \frac{1}{r} = 2 - (2 - x) = x$, so $x \in [-\frac{1}{r}, 2 - \frac{1}{r}] = A_r$. Therefore $x \in \bigcup_{r\in[1,\infty)} A_r$.

This proves that $[-1,2) \subseteq \bigcup_{r\in[1,\infty)} A_r$, so we conclude that $[-1,2) = \bigcup_{r\in[1,\infty)} A_r$.

(c) $\bigcap_{r\in[1,\infty)} A_r = [0,1]$.

Exercise 1: Recall that $\mathbf{R}^2 = \mathbf{R} \times \mathbf{R}$ is, geometrically, the set of points in the plane. For each real number r, define the subset A_r of $\mathbf{R}^2$ by $A_r = \{ (x,y) \in \mathbf{R}^2 \mid y = rx \}$.

(a) Graph A_{-1}, A_0, and A_1.

(b) Let $B = \{ (x,y) \in \mathbf{R}^2 \mid \text{ if } x = 0 \text{ then } y = 0 \}$.

(Note that B consists of all points in $\mathbf{R}^2$ except those on the positive and negative y axis.)

Prove that $\bigcup_{r\in\mathbf{R}} A_r = B$.

Hint 1: To show that $\bigcup_{r\in\mathbf{R}} A_r \subseteq B$, let $(a,b) \in \mathbf{R}^2$ and suppose $(a,b) \in \bigcup_{r\in\mathbf{R}} A_r$. To prove that $(a,b) \in B$ you need only to show that if $a = 0$ then $b = 0$.

Hint 2: To show that $B \subseteq \bigcup_{r\in\mathbf{R}} A_r$, let $(a,b) \in \mathbf{R}^2$ and suppose $(a,b) \in B$. Consider two cases, $a = 0$ and $a \neq 0$. In the case where $a \neq 0$ set $r = \frac{b}{a}$.

Theorem 1: Let $\{ A_\alpha \}_{\alpha\in\Lambda}$ be an indexed family of sets in the universal set U and let B be a set in U.

(a) $B \bigcup \Big(\bigcup_{\alpha\in\Lambda} A_\alpha \Big) = \bigcup_{\alpha\in\Lambda} (B \bigcup A_\alpha)$.

(b) $B \bigcap \Big(\bigcap_{\alpha\in\Lambda} A_\alpha \Big) = \bigcap_{\alpha\in\Lambda} (B \bigcap A_\alpha)$.

(c) $B \bigcap \Big(\bigcup_{\alpha\in\Lambda} A_\alpha \Big) = \bigcup_{\alpha\in\Lambda} (B \bigcap A_\alpha)$.

(d) $B \bigcup \Big(\bigcap_{\alpha\in\Lambda} A_\alpha \Big) = \bigcap_{\alpha\in\Lambda} (B \bigcup A_\alpha)$.

(e) $\Big(\bigcup_{\alpha\in\Lambda} A_\alpha \Big)' = \bigcap_{\alpha\in\Lambda} A'_\alpha$.

(f) $\Big(\bigcap_{\alpha\in\Lambda} A_\alpha \Big)' = \bigcup_{\alpha\in\Lambda} A'_\alpha$.

Proof of (c): To see that $B \bigcap \Big(\bigcup_{\alpha\in\Lambda} A_\alpha \Big) \subseteq \bigcup_{\alpha\in\Lambda} (B \bigcap A_\alpha)$, let $x \in U$ and assume that $x \in B \bigcap \Big(\bigcup_{\alpha\in\Lambda} A_\alpha \Big)$. Then $x \in B$ and $x \in \bigcup_{\alpha\in\Lambda} A_\alpha$. Since $x \in \bigcup_{\alpha\in\Lambda} A_\alpha$, there exists $\beta \in \Lambda$ such that $x \in A_\beta$. Therefore, $x \in B \bigcap A_\beta$. It follows that $x \in \bigcup_{\alpha\in\Lambda} (B \bigcap A_\alpha)$. This proves that $B \bigcap \Big(\bigcup_{\alpha\in\Lambda} A_\alpha \Big) \subseteq \bigcup_{\alpha\in\Lambda} (B \bigcap A_\alpha)$.

To see that $\bigcup_{\alpha\in\Lambda} (B \bigcap A_\alpha) \subseteq B \bigcap \Big(\bigcup_{\alpha\in\Lambda} A_\alpha \Big)$ let $x \in \bigcup_{\alpha\in\Lambda} (B \bigcap A_\alpha)$. Thus, there exists $\lambda \in \Lambda$ such that $x \in B \bigcap A_\lambda$. Therefore, $x \in B$ and $x \in A_\lambda$. Since $x \in A_\lambda$ it follows that $x \in \bigcup_{\alpha\in\Lambda} A_\alpha$. Consequently, $x \in B \bigcap \Big(\bigcup_{\alpha\in\Lambda} A_\alpha \Big)$ and it follows that $\bigcup_{\alpha\in\Lambda} (B \bigcap A_\alpha) \subseteq B \bigcap \Big(\bigcup_{\alpha\in\Lambda} A_\alpha \Big)$.

We conclude that $B \bigcap \Big(\bigcup_{\alpha\in\Lambda} A_\alpha \Big) = \bigcup_{\alpha\in\Lambda} (B \bigcap A_\alpha)$.

Exercise 2: Use Theorem 1 and the basic identities from Section 2.2 to prove that (with notation as in Theorem 1) $B - \bigcap_{\alpha\in\Lambda} A_\alpha = \bigcup_{\alpha\in\Lambda} (B - A_\alpha)$.

SECTION 3.3 EXERCISES:

3.3.1. For each natural number n let $A_n = [0, 3 - \frac{1}{n})$.

(a) Describe (without proof) $\bigcup_{n\in\mathbf{N}} A_n$.

(b) Describe (without proof) $\bigcap_{n\in\mathbf{N}} A_n$.

(c) If $[0,\infty)$ is the universal set then describe $\bigcup_{n\in\mathbf{N}} A'_n$ and $\bigcap_{n\in\mathbf{N}} A'_n$. [cf. Theorem 1 (e) and (f).]

3.3.2. For each positive real number α, define a subset A_α of the plane by

$$A_\alpha = \{\, (x,y) \mid x^2 + y^2 = \alpha^2 \,\}.$$

(a) Give a graphical representation for $\bigcup\{\, A_\alpha \mid \alpha \in [1,2] \,\}$.

(b) Set $B = \{\, (x,y) \mid 1 \le x^2 + y^2 \le 4 \,\}$. Prove that $\bigcup\{\, A_\alpha \mid \alpha \in [1,2] \,\} = B$. [HINT: To show that $B \subseteq \bigcup\{\, A_\alpha \mid \alpha \in [1,2] \,\}$ suppose $(a,b) \in B$. Now exhibit α such that $\alpha \in [1,2]$ and $(a,b) \in A_\alpha$

3.3.3. For each positive real number α, define a subset, A_α, of the plane by

$$A_\alpha = \{\, (x,y) \mid 0 < y \le -\frac{\alpha}{4}x^2 + \alpha \,\}.$$

Thus, A_α is the set of points in the plane that lie above the x-axis and on or under the graph of the parabola $y = -\frac{\alpha}{4}x^2 + \alpha$.
Let B be the set of points in the plane defined by

$$B = \{\, (x,y) \mid -2 < x < 2 \text{ and } y > 0 \,\}.$$

(a) Give a graphical representation of the sets A_1 and A_4 and B.

(b) Prove that

$$\bigcup_{\alpha\in\mathbf{R}^+} A_\alpha = B.$$

[**HINT**: To show that $B \subseteq \bigcup_{\alpha\in\mathbf{R}^+} A_\alpha$, let $(a,b) \in B$. Show the existence of a positive real number α such that $b = -\frac{\alpha}{4}a^2 + \alpha$.]

3.3.4. Let $\{\, A_\alpha \,\}_{\alpha\in I}$ be a nonempty collection of sets. For any set B give an elementwise proof that $B \bigcup \left(\bigcap_{\alpha\in I} A_\alpha \right) = \bigcap_{\alpha\in I}(B \bigcup A_\alpha)$.

3.3.5. Let $\{\, A_\alpha \,\}_{\alpha\in I}$ be a nonempty collection of sets. Use basic set equalities (i.e., theorems) to prove that for any set B, $B - \bigcup_{\alpha\in I} A_\alpha = \bigcap_{\alpha\in I}(B - A_\alpha)$.