CHAPTER 2: METHODS OF PROOF

Section 2.1: BASIC PROOFS WITH QUANTIFIERS

Existence Proofs

Our first goal is to prove a statement of the form $(\exists x)\, P(x)$. There are two types of existence proofs:

Constructive Proofs of Existence:

(a) One type of constructive proof is to display a specific value $x = a$ in the universal set for x and verify that $P(a)$ is true. We will focus on this method.

WARNING: Typically, finding the appropriate value, a, is the hardest part in a proof of this type and the process of finding a is often presented as the proof. In fact, the derivation of a need not be part of the proof. The proof is what we often call the "check"; that is, verifying that $P(a)$ is true.

(b) Another type of constructive proof is to show that a particular algorithm produces a value, a, such that $P(a)$ is true.

Nonconstructive Proofs of Existence involve using previous theorems, etc. that imply the existence of an a such that $P(a)$ is true without indicating how to actually produce such an a. From calculus, the Intermediate Value Theorem and the Mean Value Theorem are examples of existence theorems that can be used in this manner.

Example 1 below gives a constructive existence proof and Example 2 gives a nonconstructive existence proof.

Example 1: Prove that there exist integers m and n such that $2m + 3n = 12$.

Proof: Set $m = 3$ and $n = 2$. Then $2m + 3n = 2(3) + 3(2) = 6 + 6 = 12$.

Example 2: Set $f(x) = x^3 - 3x^2 + 2x - 4$. Prove that there exists a real number r such that $2 < r < 3$ and $f(r) = 0$.

Proof: Note that $f(2) = 2^3 - 3(2^{2)} + 2(2) - 4 = -4$ and $f(3) = 3^3 - 3(3^{2)} + 2(3) - 4 = 2$. Thus $f(2) < 0 < f(3)$. Since f is a continuous function, by the Intermediate Value Theorem, there is a real number r such that $2 < r < 3$ and $f(r) = 0$.

To review, one form for a constructive proof is as follows:

To Prove: $(\exists x \in U_x)\, P(x)$

Form of Proof:

- Let $x = a$ (where a is a specific element in U_x).
- Verify that $P(a)$ is true.

Exercise 1: Prove: There exist distinct positive integers m, n, and r such that each is a perfect square and $m = n + r$. [Definition: An integer q is a perfect square provided there exists an integer k such that $q = k^2$.]

"For All" Proofs

Our next goal is to prove a statement of the form $(\forall x \in U_x)\, P(x)$. A typical proof has the following form:

To Prove: $(\forall x \in U_x)\, P(x)$

Form of Proof:

- Let $x \in U_x$.
- If it is useful to do so, expand upon what it means for x to be in U_x.
- Give a logical argument concluding that $P(x)$ is true.

Comment: In the **construction** of the proof (the third step in the form above), you may want to both work forward from the assumption that $x \in U_x$ and backwards from the conclusion $P(x)$, but the **presentation** of the proof should begin with the assumption $x \in U_x$ and end with the conclusion that $P(x)$ is true.

Example 3: Prove: If a is an even integer and b is an odd integer, then $a + b$ is an odd integer.

NOTE: In the following proof, the italicized comments are not part of the formal proof.

Proof: Let a be an even integer and let b be an odd integer. (*Using Definitions 1 and 2 from Section 1.4, we expand on the meaning of even and odd.*) Then there exists integers k and l such that $a = 2k$ and $b = 2l + 1$. (*Now we want to conclude something about $a + b$, so let's compute it.*) Thus, $a + b = 2k + 2l + 1 = 2(k + l) + 1$. Therefore, $a + b$ is odd.

Some Common Errors in Proving $(\forall x \in U_x)\, P(x)$:

Error 1: Putting additional constraints on x beyond the assumption that $x \in U_x$. This proves only $(\exists x \in U_x)\, P(x)$.

Illustration: Prove that for every positive real number x, $x + \frac{4}{x} \geq 4$.

Proof: Let $x = 1$. Then $x + \frac{4}{x} = 1 + \frac{4}{1} = 5 > 4$.

Comment: The "proof" above showed only that 1 is in the truth set. Since the set of positive reals is an infinite set, we can never try them all.

Error 2: Assuming $P(x)$ and then concluding something that is obviously true.

Illustration: Prove that for every positive real number x, $x + \frac{4}{x} \geq 4$.

Proof: Let x be a positive real number and suppose $x + \frac{4}{x} \geq 4$. Multiplying by x gives $x^2 + x \geq 4x$. Now subtracting $4x$ from both sides we get $x^2 - 4x + 4 \geq 0$ which is the same as $(x-2)^2 \geq 0$, which is always true.

Comment: What was proved above is that for all positive real numbers x, if $x + \frac{4}{x} \geq 4$, then $(x-2)^2 \geq 0$, a conclusion that is true in any case. In essence, this is the proof done backwards.
But the above "proof" does help us to construct a valid proof. In the presentation, we just need to reverse the steps so we begin with that which is obviously true and conclude with $P(x)$.

Illustration of a Valid Proof : Prove that for every positive real number x, $x + \frac{4}{x} \geq 4$.

Proof: Let x be a positive real number. Then clearly $(x-2)^2 \geq 0$ since the square of a real number is never negative. Expanding gives $x^2 - 4x + 4 \geq 0$. By assumption x is positive, so dividing by x preserves the inequality and gives $x - 4 + \frac{4}{x} \geq 0$. Finally, adding 4 to both sides gives $x + \frac{4}{x} \geq 4$.

Exercise 2: Prove that the sum of any two rational numbers is again rational.
NOTE: The statement to be proved can be stated symbolically as follows, where $\mathbf{Q}$ denotes the set of all rational numbers:

$$(\forall r, s \in \mathbf{Q}),\, r + s \in \mathbf{Q}.$$

"For All" Proofs; Division into Cases

Sometimes in proving $(\forall x \in U_x)\, P(x)$ we cannot find a proof that applies at once to all the elements in U_x. In such instances, it is often convenient to divide the proof into cases. That means that we partition U_x into subsets and give a separate proof for each subset.

Example 4: Prove that any two consecutive integers have opposite parity. (That is, if one is even the other is odd and vice versa.)

Proof: Let m and n be consecutive integers with $m < n$. Then $n = m + 1$.

Case 1: Suppose m is even. Then there exists an integer k such that $m = 2k$. Therefore, $n = m + 1 = 2k + 1$, so n is odd.

Case 2: Suppose m is odd. Then there exists an integer k such that $m = 2k + 1$. Therefore, $n = m + 1 = (2k + 1) + 1 = 2k + 2 = 2(k + 1)$, so it follows that n is even.

SECTION 2.1: EXERCISES

2.1.1. Prove: There exists an even integer n that can be written in two different ways as a sum of two distinct primes. [CAUTION: 1 is not a prime. Also note that something like $8 = 3 + 5 = 5 + 3$ does not count since in both sums the two primes are the same.]

2.1.2. Disprove (that is, prove the negation): For every positive integer n, $3^n + 2$ is prime.

2.1.3. Disprove: For all integers r, m, and n, if r divides mn then either r divides m or r divides n.

2.1.4. Prove: For all odd integers a and b, ab is also odd.

2.1.5. Prove: For every integer n, the integer $n^2 + n$ is even. [HINT: Take cases; n even and n odd.]

2.1.6. Prove: For every positive integer n, $n^2 + 4n + 3$ is not a prime.

2.1.7. Prove: If n is the product of any four consecutive integers then $n + 1$ is a perfect square.

Section 2.2. BASIC PROOFS WITH MIXED QUANTIFIERS

Proving Statements of the Form $(\exists x)(\forall y)\, P(x, y)$.

The proof of $(\exists x \in U_x)(\forall y \in U_y)\, P(x, y)$ is firstly an existence proof and secondly a "for all" proof. If possible, we set $x = a$, where a is a specific element in U_x. Next we let y be an arbitrary (variable) element in U_y and prove that $P(a, y)$ is true. This is summarized as follows:

To Prove: $(\exists x \in U_x)(\forall y \in U_y)\, P(x, y)$

Form of Proof:

- Set $x = a$ where a is a specific element in U_x.
- Let y be an arbitrary (that is, variable) element in U_y.
- If useful, expand on what $y \in U_y$ means.
- Verify that $P(a, y)$ is true.

Example 1: Prove that there exists a real number x such that for every real number y, $xy - 3x - 3y + 12 = y$.

Construction of the Proof: Working backwards, we want to find a specific real number x such that $xy - 3x - 3y + 12 = y$ for every real number y.
$xy - 3x - 3y + 12 = y$ Adding $-3y + 12$ to both sides gives
$xy - 3x = 4y - 12$ Factor x out on the left side
$x(y - 3) = 4y - 12$ Divide by $y - 3$
$x = \frac{4y-12}{y-3} = \frac{4(y-3)}{y-3} = 4$

Proof: Set $x = 4$ and let y be a real number. Then
$xy - 3x - 3y + 12 = 4y - 3(4) - 3y + 12 = y$.

Exercise 1: Prove that there exists a 2×2 matrix A such that for every 2×2 matrix B, $AB = 3B$.

Proving Statements of the Form $(\forall x)(\exists y)\, P(x, y)$

The proof of $(\forall x \in U_x)(\exists y \in U_y)\, P(x, y)$ is firstly a "for all" proof and secondly an existence proof. Let x be an arbitrary (variable) element in U_x. Now find y, usually expressed in terms of x – say $y = g(x)$ – and prove that $P(x, g(x))$ is true. This is summarized as follows:

To Prove: $(\forall x \in U_x)(\exists y \in U_y)\, P(x, y)$

Form of Proof:

- Let x be an arbitrary (i.e., variable) element in U_x.
- If it is helpful, expand on what $x \in U_x$ means.
- Based on construction work done before you begin writing the proof, define y in terms of x – say $y = g(x)$.
- Verify that $P(x, g(x))$ is true.

Example 2: Prove that for every real number x there exists a real number y such that $x^2y + 2x = x$.

Construction of the Proof: For arbitrary (variable) x we want to find y, in terms of x, so that $x^2y + 2x = x$. Solving this equation for y gives $y = \frac{-1}{x}$. Clearly, this choice of y is not defined for $x = 0$, but we can see that any choice of y works if $x = 0$. Thus, we divide the proof into cases.

Proof: Let x be a real number.
Case 1: If $x \neq 0$ then set $y = \frac{-1}{x}$. Then $x^2y + 2x = x^2(\frac{-1}{x}) + 2x = -x + 2x = x$.
Case 2: If $x = 0$ set $y = 1$ (or any other choice of y). Then $x^2y + 2x = (0)1 + 0 = 0 = x$.

Exercise 2: Prove that for every real number $y > -2$ there exists a real number x such that $y = 3e^x - 2$.

[NOTE: That $y > -2$ is a glaring restriction on y, so in your proof you should note exactly where that restriction is required.]

As Case 2 in the proof of Example 2 above illustrates, sometimes in proving $(\forall x \in U_x)(\exists y \in U_y)\, P(x, y)$ there are multiple choices for y. The following example demonstrates this.

Example 3: Prove that for every real number x there exist real numbers y and z such that $2x - 3y + 4z = 12$.

Construction of the Proof: For arbitrary x we want to find specific y and z, perhaps expressed in terms of x, such that $2x - 3y + 4z = 12$. From this equation we get $-3y + 4z = 12 - 2x$. But there are infinitely many choices for y and z. For instance, set $z = 0$. Then we get $-3y = 12 - 2x$, so $y = -4 + \frac{2}{3}x$.

Proof: Let x be a real number. Set $y = -4 + \frac{2}{3}x$ and set $z = 0$. Then $2x - 3y + 4z = 2x - 3(-4 + \frac{2}{3}x) + 4(0) = 2x + 12 - 2x + 0 = 12$.

Proving Existence and Uniqueness

We must frequently prove statements of the form "there exists a unique x such that $P(x)$." To say that x is unique does not mean that x has purple hair. It means, rather,that there is only one x such that $P(x)$ is true. We will denote this symbolically as $(\exists\,!\, x \in U_x)\, P(x)$. Thus "!" symbolizes the uniqueness of x.
We present two common approaches to proving uniqueness.

Method 1 for Proving $(\exists\,!\, x \in U_x)\, P(x)$

- First prove the existence of $x = a$ such that $P(a)$ is true.
- To prove uniqueness,
  - let a_1 and a_2 be (variable) elements in U_x.
  - Assume that both $P(a_1)$ and $P(a_2)$ are true.
  - Prove that $a_1 = a_2$.

Method 2 for Proving $(\exists\,!\, x \in U_x)\, P(x)$

- Let x be a (variable) element in U_x.
- Suppose $P(x)$ is true.
- Show that the assumption that $P(x)$ is true leads to one, and only one, value $x = a$.
- Verify that $P(a)$ is indeed true.

Thus, this method has the advantage of proving both existence and uniqueness at once.

In Example 4 we will illustrate both Method 1 and Method 2.

Example 4: Prove that there exists a unique real number x such that $\ln x = 2$.

First Proof (Method 1): To prove existence, set $x = e^2$. Then $\ln x = \ln(e^2) = 2$.

To see that x is unique, suppose a and b are real numbers such that $\ln a = 2$ and $\ln b = 2$. Then $\ln a = \ln b$, so $e^{\ln a} = e^{\ln b}$. Therefore $a = b$.

Second Proof (Method 2): Suppose that x is a real number such that $\ln x = 2$. Then $e^{\ln x} = e^2$, so $x = e^2$. Thus, x is uniquely determined, and indeed, if we set $x = e^2$, we get $\ln x = \ln(e^2) = 2$.

Exercise 3: Prove that for every real number $y > -2$ there exists a unique real number x such that $y = 3e^x - 2$.

NOTE: Existence was proved in Exercise 2 above.

SECTION 2.2. EXERCISES

2.2.1. Prove: There exists a unique integer m such that for every integer n, $mn + 2m + 2n + 2 = n$.

2.2.2. Prove: There exists a unique integer m such that for every integer n, $mn + m + n + 1 = 0$.

2.2.3. Prove: For every integer n there exists a unique integer m such that $2m + 8n = 6$.

2.2.4. Prove: For every integer d there exists integers a, b, and c such that $-a + 3b - 2bc + 4 = d$.

2.2.5. Prove: For all integers a, b, c, and d with $a \neq c$ and $ad - bc \neq 0$, there exists a unique rational number r such that

$$\frac{ar + b}{cr + d} = 1.$$

2.2.6. Prove: For every integer n there exists integers a, b, c, and d such that $ab - cd = n$.

2.2.7. Prove: For every positive integer n there exists an odd integer m such that $2^{2n} + m$ is a perfect square.

SECTION 2.3: IMPLICATIONS

An implication of the general form $(\forall x \in U_x)\,(P(x) \to Q(x))$ is one of the most frequently occurring forms of a mathematical statement. In Sections 2.3 and 2.4 we will introduce three methods for proving such a statement. These are

- direct proofs,
- contrapositive proofs,
- and proofs by contradiction.

Direct Proofs of $(\forall x \in U_x)\,(P(x) \to Q(x))$

Recall the truth table for $P \to Q$. If P is false then $P \to Q$ is true. Indeed, $P \to Q$ is false only when P is true and Q is false. Thus, to prove that $P \to Q$ is a true statement, we need only to show that this one case cannot happen; that is, we must show that whenever P is true, then Q is also true. Therefore, we begin a direct proof of $P \to Q$ by assuming that P is true. In doing so, we are not asserting that P is, in fact, true. We are merely considering the case in which P is true, since the other cases (i.e., when P is false) need not be considered. We then proceed to argue that in the case when P is true, Q is necessarily true also.

The statement $(\forall x \in U_x)\,(P(x) \to Q(x))$ is first a "for all" statement, so our previous methods apply.

To Prove: $(\forall x \in U_x)\,(P(x) \to Q(x))$

Form of Proof:

- Let x be arbitrary (variable) in U_x.
- If useful, expand on what $x \in U_x$ means.
- Assume $P(x)$.
- If useful expand on the assumption $P(x)$.
- Give a logical argument the concludes that $Q(x)$ is true.

Construction:

The last step in the above form is the heart of the proof. You must argue from the assumptions to the conclusion that $Q(x)$ is true. The key is to focus on the desired conclusion, $Q(x)$. If you can continue from where you are to the conclusion that $Q(x)$ is true then do so. If there is a common procedure for proving $Q(x)$, try using it. Sometimes, it helps to reverse directions and work from $Q(x)$ backwards to $P(x)$. But in the presentation, proceed from the assumption that $P(x)$ is true to the conclusion that $Q(x)$ is true.

Example 1: Prove: For all integers a, b, and c if a divides b and b divides c then a divides c.

Construction: If $P(x, y)$ is the statement that "x divides y", then our given statement has symbolic form

$$(\forall a, b, c \in \mathbf{Z}) \left[\Big(P(a, b) \wedge P(b, c)\Big) \rightarrow P(a, c)\right].$$

- Let a, b, and c be integers. (*Nothing to expand on here.*)
- Assume that a divides b and b divides c. (*Use the definition of "divides" to expand this. Note that this will introduce new variables.*)
- Then there exist integers m and n such that $b = am$ and $c = bn$. (*Now focus on the desired conclusion.*)
- We want to prove that a divides c, so we need to verify that there exists an integer q such that $c = aq$. But we have $c = bn = (am)n = a(mn)$, so $q = mn$ works.

Proof: Let a, b, and c be integers. Assume that a divides b and b divides c. Then there exist integers m and n such that $b = am$ and $c = bn$. If we set $q = mn$ then we see that $c = bn = (am)n = a(mn) = aq$. Thus, a divides c.

Example 2: Prove that for every real number x, if $x \neq 0$ and $x \neq 3$, then $1 + \frac{9}{x^2} > \frac{6}{x}$.

Construction:
$P(x)$: $x \neq 0$ $\quad$ $Q(x)$: $x \neq 3$ $\quad$ $R(x)$: $1 + \frac{9}{x^2} > \frac{6}{x}$
Form: $(\forall x) \left[\Big(P(x) \wedge Q(x)\Big) \rightarrow R(x)\right]$
Let x be a real number. (*Nothing to expand on.*). Suppose $x \neq 0$ and $x \neq 3$. (*Again, nothing to expand on.*) Now let's focus on the desired conclusion $1 + \frac{9}{x^2} > \frac{6}{x}$. Since the assumptions don't give us much direction, let's work backwards from the conclusion.
$1 + \frac{9}{x^2} > \frac{6}{x}$ Multiply by x^2. (why is this ok?)
$x^2 + 9 > 6x$ Subtract 6x from both sides
$x^2 - 6x + 9 > 0$ Factor
$(x - 3)^2 > 0$ This is true if $x \neq 3$.

Proof: Let x be a real number. Suppose $x \neq 0$ and $x \neq 3$. Since $x \neq 3$, $x - 3 \neq 0$, so $(x - 3)^2 > 0$. Expanding gives $x^2 - 6x + 9 > 0$ and adding $6x$ to both sides gives $x^2 + 9 > 6x$. Since $x \neq 0$, x^2 is positive. Thus, division by x^2 preserves the inequality and gives $1 + \frac{9}{x^2} > \frac{6}{x}$.

Exercise 1: Let a, b, c, m, and n be integers. Prove that if a divides both b and c then a divides $mb + nc$.

Contrapositive Proofs of $(\forall x \in U_x)\,(P(x) \to Q(x))$

Recall that an implication, $P \to Q$, and its contrapositive, $\sim Q \to \sim P$, are logically equivalent. Thus, to prove $(\forall x \in U_x)\,(P(x) \to Q(x))$, it suffices to prove $(\forall x \in U_x)\,(\sim Q(x) \to \sim P(x))$. Thus, a contrapositive proof proceeds just as a direct proof with the exception that we assume $\sim Q(x)$ and conclude $\sim P(x)$. A contrapositive proof has the following form.

To Prove: $(\forall x \in U_x)\,(P(x) \to Q(x))$

Form of Proof:

- State that the proof is by contrapositive.
- Let x be an arbitrary (variable) element in U_x.
- If useful, expand on what $x \in U_x$ means.
- Assume $\sim Q(x)$.
- If useful expand on the assumption $\sim Q(x)$.
- Give a logical argument the concludes that $\sim P(x)$ is true.

Example 3: Prove that for all integers n, if n^2 is even then n is even.

Construction:
If $P(x)$ is the statement "x is even" then the given statement has the form $(\forall n \in \mathbf{Z})\,\big(P(n^2) \to P(n)\big)$.

Let's first try the direct approach. Let n be an integer. Suppose n^2 is even. Then there exists an integer k such that $n^2 = 2k$. We are interested in n. This gives $n = \sqrt{2k}$. Now what??

If we prove the contrapositive, we will begin with an assumption about n and try to reach a conclusion about n^2. That seems more natural, so lets' try again. In fact, this works so well, we move directly to the proof.

Proof: The proof is by contrapositive. Let n be an integer. Suppose n is not even; that is, suppose that n is odd. Then there exists an integer k such that $n = 2k + 1$. Therefore, $n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$, so it follows that n^2 is also odd.

Example 4: Use Example 3 to give a direct proof that for every integer n, if n^2 is even then n^2 is divisible by 4.

Proof: Let n be an integer. Suppose that n^2 is even. By Example 3, n is even, so there exists an integer k such that $n = 2k$. But then $n^2 = (2k)^2 = 4k^2$, so n^2 is divisible by 4.

Exercise 2: Prove that for a matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ in $M_{3\times 3}(\mathbf{R})$, if $A^3 \neq O$ then one of the entries a_{11}, a_{12}, a_{13}, a_{22}, a_{23}, a_{33} is not zero.

SECTION 2.3: EXERCISES

Direct Proofs

2.3.1. Prove that if a and b are integers such that a divides b, then a^2 divides b^2.

2.3.2. Prove that if A is an $n \times n$ real matrix such that $A^3 = A$ then $\det(A)$ must equal 1, -1, or 0.

2.3.3. Prove that if m and n are perfect squares then $m + n + 2\sqrt{mn}$ is also a perfect square.

2.3.4. (a) Let $f(x) = 2x + 4$. Prove that for every positive real number ϵ there exists a positive real number δ such that for all real numbers a and b, if $|a - b| < \delta$ then $|f(a) - f(b)| < \epsilon$. [NOTE: Before writing the proof you must find δ, likely in terms of ϵ. To do so, work backwards from $|f(a) - f(b)| < \epsilon$.]

(b) Let $f(x) = 2x^2 + 4$. Prove that for all real numbers a and b and for every positive real number ϵ there exists a positive real number δ such that if $|a - b| < \delta$ then $|f(a) - f(b)| < \epsilon$. [NOTE: (b) has a different form than (a). In this case δ is likely a function of which variables?]

2.3.5. In each of (a) – (c), find the error in the "proof" that for all real numbers r and s, if r and s are rational numbers then $r + s$ is a rational number.

(a) **Proof**: Let $r = \frac{1}{2}$ and $s = \frac{1}{3}$. Then $r + s = \frac{1}{2} + \frac{1}{3} = \frac{3}{6} + \frac{2}{6} = \frac{5}{6}$, so $r + s$ is a rational number.

(b) **Proof**: Let r and s be rational numbers. Then by the definition of a rational number, there exist integers m and n such that $n \neq 0$ and $r = \frac{m}{n}$. Likewise, there exists integers m and n such that $n \neq 0$ and $s = \frac{m}{n}$. Therefore, $r + s = \frac{m}{n} + \frac{m}{n} = \frac{2m}{n}$. It follows that $r + s$ is a rational number.

(c) **Proof**: Let r and s be rational numbers. Then by the definition of a rational number, there exist integers a, b, c, and d such that $b \neq 0$, $d \neq 0$, $r = \frac{a}{b}$, and $s = \frac{c}{d}$. Therefore, $r + s = \frac{a}{b} + \frac{c}{d}$ is a sum of two fractions. Since a sum of fractions is again a fraction, $r + s$ is a rational number.

Contrapositive Proofs

INSTRUCTIONS: In each of 2.3.5 – 2.3.7, prove the given statement contrapositively.

2.3.6. Prove that if a is an odd integer then the equation $x^2 + x - a = 0$ has no integer solution. [HINT: Use Exercise 2.1.5]

2.3.7. Let a, b, and c be consecutive integers with $a < b < c$. Prove that if $a \neq -1$ and $a \neq 3$ then $a^2 + b^2 \neq c^2$.

2.3.8. (a) Let n, a, b be integers, where $n > 1$. Prove that if $n = ab$ then either $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$.

Theorem for 2.3.8.(b): If a is an integer such that $a > 1$ then a is divisible by a prime integer.

(b) Let n be an integer with $n > 1$. Give a direct proof that if n is not prime then there exists a prime integer p such that p divides n and $p \leq \sqrt{n}$. [HINT: Recall that if n is not prime then n is a composite; that is, there exist positive integers a and b such that $1 < a < n$, $1 < b < n$, and $n = ab$. Now use (a), the given Theorem, and Example 1.]

(c) Use (b) to determine if 103 is prime. What primes must be tested as possible divisors of 103?

SECTION 2.4: PROOFS BY CONTRADICTION AND EQUIVALENT STATEMENTS

Proofs by Contradiction

A proof by contradiction has the following general form:

To Prove: P
Form of Proof:

- State that the proof is by contradiction
- Assume $\sim P$.
- Work from the assumption, $\sim P$ until you conclude $R \wedge \sim R$ for some statement R.
- State that a contradiction has been reached, so conclude that P is true.

Comment: In practice, in the second to last step of the form, we often just conclude, say $\sim R$, where R is already known to be true. For example, if we conclude that $1 < 0$ then we have reached a contradiction since we already know that $1 \geq 0$.

The logic of the above method is as follows:
- In the proof, you have shown that $\sim P \rightarrow (R \wedge \sim R)$ is a true statement.
- The statement $R \wedge \sim R$ is clearly false.
- Therefore, $\sim P$ must also be false, so P must be true.

Example 1: Prove that for every positive real number x,

$$\frac{x}{x+1} < \frac{x+1}{x+2}.$$

Construction: If U_x is the set of all positive real numbers and $P(x)$ denotes the statement

$$\frac{x}{x+1} < \frac{x+1}{x+2}$$

then the statement to be proved has the form $(\forall x \in U_x)\, P(x)$. Thus, a proof by contradiction begins by assuming the negation, $(\exists x \in U_x)(\sim\ P(x))$.

Proof: The proof is by contradiction, so assume that there exists a real number x such that

$$\frac{x}{x+1} \geq \frac{x+1}{x+2}.$$

Since x is positive, both $x+1$ and $x+2$ are positive. Hence, $(x+1)(x+2)$ is positive, so we can multiply both sides of the inequality

$$\frac{x}{x+1} \geq \frac{x+1}{x+2}$$

by $(x+1)(x+2)$ and still preserve the inequality. This gives $x(x+2) \geq (x+1)^2$. Expanding both sides results in $x^2+2x \geq x^2+2x+1$. Subtracting x^2+2x from both sides leaves $0 \geq 1$, a contradiction (to the known fact that $0 < 1$), so we conclude that for every positive real number x,

$$\frac{x}{x+1} < \frac{x+1}{x+2}.$$

Before stating the next exercise, we first review some relevant theorems from calculus.

Theorem A: If $F(x) = G(x)$ for all x except possibly at $x = a$ then $\lim_{x\to a} G(x) = \lim_{x\to a} F(x)$ provided the latter limit exists.

Theorem B: If $\lim_{x\to a} F(x)$ and $\lim_{x\to a} G(x)$ both exists then $\lim_{x\to a} F(x)G(x) = \lim_{x\to a} F(x) \lim_{x\to a} G(x)$.

Exercise 1: Prove by contradiction that $\lim_{x\to 0}(\cos x)/x$ does not exist.

HINT: Suppose the limit does exist, say $\lim_{x\to 0}(\cos x)/x = L$. Now use the Theorem A above to get one value for $\lim_{x\to 0} x(\cos x)/x$, then use Theorem B to get another value.

Proving $(\forall x \in U_x)\left(P(x) \to Q(x)\right)$ by Contradiction

The proof of an implication $(\forall x \in U_x)\left(P(x) \to Q(x)\right)$ by contradiction takes the following form:

To Prove: $(\forall x \in U_x)\left(P(x) \to Q(x)\right)$

Form of Proof:

- State that the proof is by contradiction.
- Assume $(\exists x \in U_x)\left(P(x) \wedge \sim Q(x)\right)$.
- Work from the assumption of the previous step until you conclude $R \wedge \sim R$ for some statement R.
- State that a contradiction has been reached so we may conclude that $(\forall x \in U_x)\left(P(x) \to Q(x)\right)$.

Example 2: Prove by contradiction: If

$$f(x) = \frac{2x+3}{x+2}$$

then for every $x \in \mathbf{R}$, $f(x) \neq 2$.

Construction: Since our proof is by contradiction, we will need to negate that which is to be proved. The symbolic form of the given statement is:

$$\Big(f(x) = \frac{2x+3}{x+2}\Big) \to (\forall x \in \mathbf{R})\,(f(x) \neq 2).$$

Therefore, the negation has symbolic form

$$(f(x) = \frac{2x+3}{x+2}) \wedge (\exists x \in \mathbf{R})\,(f(x) = 2).$$

Proof: The proof is by contradiction, so assume that

$$f(x) = \frac{2x+3}{x+2}$$

and assume that there exists a real number x such that $f(x) = 2$. This gives

$$\frac{2x+3}{x+2} = 2$$

for some real number x. Multiplying both sides by $x+2$ gives $2x+3 = 2(x+2) = 2x+4$. Subtracting $2x$ from both sides gives $3 = 4$, a contradiction. We conclude, therefore, that for every $x \in \mathbf{R}$, $f(x) \neq 2$.

Exercise 2: Prove by contradiction: For every real number x, if $x > 0$ then $x + \frac{1}{x} \geq 2$.

Disadvantages of Proofs by Contradiction:

- We don't know what we are trying to prove. We are trying to arrive at a contradiction, but we don't know what it will be.
- Proofs by contradiction are usually not constructive so they leave us with little intuition about why the result is true.
- An error can lead to a contradiction and the false sense of having successfully completed a proof.

Equivalencies

Recall that statements P and Q are equivalent, written $P \leftrightarrow Q$, provided they have the same truth value; that is, both are true or both are false.

Following are some of the most common English wordings that translate symbolically to $P \leftrightarrow Q$.

- P is equivalent to Q.
- P if and only if Q.
- P is necessary and sufficient for Q.

Comment: Equivalence, denoted by "$\leftrightarrow$", and logical equivalence, denoted by "$\Leftrightarrow$", are not the same. Logical equivalence refers to statement forms, not actual statements. If actual statements are substituted into logically equivalent forms, the resulting statements will be equivalent, with no proof required. For example, $P \to Q$ and $\sim Q \to \sim P$ are logically equivalent forms, so when specific statements are substituted for P and Q the result is equivalent statements.

On the other hand, two statements may well be equivalent when their underlying forms are not logically equivalent.

Theorem 1: The statement forms $P \leftrightarrow Q$ and $(P \to Q) \wedge (Q \to P)$ are logically equivalent.

Proof:

P	Q	$P \leftrightarrow Q$	$P \to Q$	$Q \to P$	$(P \to Q) \wedge (Q \to P)$
T	T	T	T	T	T
T	F	F	F	T	F
F	T	F	T	F	F
F	F	T	T	T	T

It follows from the preceding theorem that we can prove $P \leftrightarrow Q$ by proving both $P \to Q$ and $Q \to P$. Thus, a proof of $P \leftrightarrow Q$ is actually two proofs and for each part we may use whatever method works best; a direct proof, a contrapositive proof, or a proof by contradiction.

Example 3: For real numbers a, b, and c, set $A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$. Prove that $A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ if and only if $a = 0$ and $c = 0$.

Construction: We will use the following assignments:

$U_a = U_b = U_c = \mathbf{R}$

U_A: the set of all matrices of the form $\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$.

$P(A)$: $A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

$Q(x)$: $x = 0$.

Thus, the statement to be proved has symbolic form

$$(\forall a, b, c, A)\left[P(A) \leftrightarrow \Big(Q(a) \wedge Q(c)\Big)\right].$$

Thus, we must prove both

$$(\forall a, b, c, A)\left[P(A) \rightarrow \Big(Q(a) \wedge Q(c)\Big)\right]$$

and

$$(\forall a, b, c, A)\left[\Big(Q(a) \wedge Q(c)\Big) \rightarrow P(A)\right]$$

Proof: Let a, b and c be real numbers and let $A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$.

First, suppose $A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Then

$$A^2 = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} = \begin{bmatrix} a^2 & ab+bc \\ 0 & c^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

It follows that $a^2 = 0$ and $c^2 = 0$. Therefore, $a = 0$ and $c = 0$.

Now, in the other direction, assume that $a = 0$ and $c = 0$. Then

$$A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} = \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} \text{ so } A^2 = \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

In the following exercise, $\mathbf{R}^2$ denotes the set of all ordered pairs of real numbers. Geometrically, $\mathbf{R}^2$ is the set of points in the plane. Thus, an element of $\mathbf{R}^2$ has the form (x, y), where x and y are real numbers.

The notation $f : \mathbf{R}^2 \to \mathbf{R}$ means that f is a function that corresponds points in $\mathbf{R}^2$ to real numbers. For instance, if $f(x, y) = \ln(x^2+1) - y^3$, then $f(0, 2) = \ln(1) - 2^3 = -8$.

Exercise 3: Let $f : \mathbf{R}^2 \to \mathbf{R}$ be defined by $f(x, y) = \ln(x^2+1) - y^3$.

Prove: For all $(x, y) \in \mathbf{R}^2$, $f(x, y) = 1$ if and only if $x = \pm\sqrt{e^{y^3+1} - 1}$.

SECTION 2.4: EXERCISES

Proofs by Contradiction

2.4.1. Prove by contradiction: If a, b, and c are consecutive integers such that $a < b < c$ then $a^3 + b^3 \neq c^3$.

2.4.2. Prove by contradiction: For all integers a, b, and c, if $a^2 + b^2 = c^2$ then either a is even or b is even. [**HINT**: Use Example 4 of Section 2.3; in particular, show that c^2 is even but not divisible by 4.]

2.4.3. Use the method of contradiction to disprove: For all integers a and b there exist integers m and n such that $a = m + n$ and $b = m - n$.

2.4.4. Let p_1, p_2,...,p_k be prime integers. Prove by contradiction that if $n = p_1 p_2 \cdots p_k + 1$ then for every i, $i = 1, 2, \ldots, k$, p_i does not divide n. [**NOTE**: 1 is not divisible by a prime, so to arrive at the conclusion that 1 is divisible by some prime would be a contradiction.]

2.4.5. Prove by contradiction that there are infinitely many primes. [**HINT**: Use Exercise 2.4.4 and the theorem given in Exercise 2.3.8(b).]

Proofs of Equivalence

2.4.6. Let a and b be nonzero integers. Prove that a divides b and b divides a if and only if $a = \pm b$.

2.4.7. Prove that if y is a real number, then there exists a real number x such that

$$\frac{2x-1}{x-3} = y$$

if and only if $y \neq 2$.

2.4.8. Prove that for all nonzero real numbers x and y, we have $\frac{x}{y} + \frac{y}{x} \geq 2$ if and only if either $x > 0$ and $y > 0$ or $x < 0$ and $y < 0$.

2.4.9. REVIEW:

- Let A be a 2×2 matrix. Recall that an inverse for A is a 2×2 matrix A^{-1} such that $AA^{-1} = I_2$ and $A^{-1}A = I_2$, where $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is the 2×2 identity matrix.
- If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then the determinant of A is $\det(A) = ad - bc$.
- For matrices A and B, $\det(AB) = \det(A)\det(B)$.

(a) Prove that a 2×2 matrix A has an inverse if and only if $\det(A) \neq 0$. [**HINT**: In the case when $\det(A) \neq 0$, consider the matrix $\begin{bmatrix} d/\det(A) & -b/\det(A) \\ -c/\det(A) & a/\det(A) \end{bmatrix}$.]

(b) Let C and D be 2×2 matrices. Use (a) and the given matrix properties to prove that $CD = I_2$ if and only if $DC = I_2$.

Discussion: First, recall that matrix multiplication is not commutative, so if, for example, $CD = I_2$ we cannot automatically conclude that $DC = I_2$. But, by the definition in the first bullet above, the inverse of a matrix must work on both sides.

HINT: Show that $\det(C) \neq 0$ and, hence, C^{-1} exists by (a). Now argue that $D = C^{-1}$.

SECTION 2.5: SUMMARY: FORM!! FORM!! FORM!!

Arriving at a valid proof involves some combination of

- proper form,
- knowledge of subject, and
- ingenuity.

Just as an accomplished writer uses proper language and grammer without much thought, form becomes essentially automatic for an experienced proof writer. For this course, however, form is the focus. In many of the proofs you encounter later in the course, proper form will be 80 to 90 percent of a correct proof.

Procedure for Writing a Proof

Following is a suggested four step process for writing a proof. In the following material we will examine each of the first, third, and fourth steps more closely. Typically, only the presentation is submitted to the reader.

Steps in Writing a Proof

STEP 1: Give a symbolic representation of the statement to be proved and outline the form of the proof.

STEP 2: Do a preliminary construction of the proof.

- Solve all existence assertions.
- Derive all arguments that proceed from some assumption P to some conclusion Q.

(This is scratch work so you can start with P and work to Q or start with Q and work backwards to P.)

STEP 3: Present your Proof

STEP 4: Check your proof against the following guidelines.

1. The beginning of the proof is clearly marked with the word **Proof**.
2. The appropriate form is followed.
3. The reader is informed of special proof techniques (contrapositive, contradiction, induction) to be employed.
4. All variables that appear in the proof are introduced in the appropriate context within the proof.
5. The proof is written in complete English sentences.

6. All assumptions used in the proof are stated in the proof.

7. All logical arguments begin with the appropriate assumption, end with the desired conclusion and provide sufficient detail to lead the reader through the argument.

We now explore Steps 2 and 4 of the process in more detail.

Basic Forms for Writing Proofs

We collect here all the basic argument forms given in earlier sections:

Argument Form 1:
To Prove: $(\exists x \in U_x)\, P(x)$
Form of Proof:

- Let $x = a$ (where a is a specific element in U_x).
- Verify that $P(a)$ is true.

Argument Form 2:
To Prove: $(\forall x \in U_x)\, P(x)$
Form of Proof:

- Let $x \in U_x$.
- If it is useful to do so, expand upon what it means for x to be in U_x.
- Give a logical argument concluding that $P(x)$ is true.

Argument Form 3:
To Prove: $(\exists x \in U_x)(\forall y \in U_y)\, P(x, y)$

Form of Proof:

- Set $x = a$ where a is a specific element in U_x.
- Let y be an arbitrary (that is, variable) element in U_y.
- If useful, expand on what $y \in U_y$ means.
- Verify that $P(a, y)$ is true.

Argument Form 4:

To Prove: $(\forall x \in U_x)(\exists y \in U_y)\, P(x, y)$

Form of Proof:

- Let x be an arbitrary (i.e., variable) element in U_x.
- If it is helpful, expand on what $x \in U_x$ means.
- Based on construction work done before you begin writing the proof, define y in terms of x – say $y = g(x)$.
- Verify that $P(x, g(x))$ is true.

Argument Form 5:

Method 1 for Proving $(\exists!\, x \in U_x)\, P(x)$

- First prove the existence of $x = a$ such that $P(a)$ is true.
- To prove uniqueness,
  - let a_1 and a_2 be (variable) elements in U_x.
  - Assume that both $P(a_1)$ and $P(a_2)$ are true.
  - Prove that $a_1 = a_2$.

Method 2 for Proving $(\exists!\, x \in U_x)\, P(x)$

- Let x be a (variable) element in U_x.
- Suppose $P(x)$ is true.
- Show that the assumption that $P(x)$ is true leads to one, and only one, value $x = a$.
- Verify that $P(a)$ is indeed true.

Argument Form 6:

To Prove: $(\forall x \in U_x)\,(P(x) \to Q(x))$

Form of Direct Proof:

- Let x be arbitrary (variable) in U_x.
- If useful, expand on what $x \in U_x$ means.
- Assume $P(x)$.
- If useful expand on the assumption $P(x)$.
- Give a logical argument that ends with the conclusion that $Q(x)$ is true.

Argument Form 7:

To Prove: $(\forall x \in U_x)\,(P(x) \to Q(x))$

Form of a Contrapositive Proof:

- State that the proof is by contrapositive.
- Let x be an arbitrary (variable) element in U_x.
- If useful, expand on what $x \in U_x$ means.
- Assume $\sim Q(x)$.
- If useful expand on the assumption $\sim Q(x)$.
- Give a logical argument that ends with the conclusion that $\sim P(x)$ is true.

Argument Form 8:

To Prove: P

Form of a Proof by Contradiction:

- State that the proof is by contradiction
- Assume $\sim P$.
- Give a logical argument that ends with the conclusion that $R \wedge \sim R$ for some statement R.
- State that a contradiction has been reached, so conclude that P is true.

Argument Form 9:

To Prove: $(\forall x \in U_x)\left(P(x) \to Q(x)\right)$

Form of a Proof by Contradiction:

- State that the proof is by contradiction.
- Assume $(\exists x \in U_x)\left(P(x) \wedge \sim Q(x)\right)$.
- Give a logical argument that ends with the conclusion that $R \wedge \sim R$ for some statement R.
- State that a contradiction has been reached so we may conclude that $(\forall x \in U_x)\left(P(x) \to Q(x)\right)$.

Example 1: As an illustration, let's combine some of these forms into a single statement.

To Prove: $(\forall x \in U_x)\left[(\exists y \in U_y)\, P(x,y) \to (\forall z \in U_z)\, Q(x,z)\right]$

Form of Proof

- Let x be arbitrary (variable) in U_x.
- If it helps, expand on what it means to be in U_x.
- Assume $(\exists y \in U_y)\, P(x,y)$.
- If it helps, expand on this assumption above.
- Let z be arbitrary (variable) in U_z.
- If it helps, expand on what it means to be in U_z.
- Present a logical sequence of steps that end with the conclusion that $Q(x,y,z)$ is true.

Exercise 1: Outline the form of a contrapositive proof of the symbolic statement given in Example 1.

Exercise 2: Outline the form of a proof by contradiction of the symbolic statement given in Example 1.

Preliminary Construction of the Proof

This is that part of the proof that requires some combination of ingenuity and a knowledge of the subject matter. As such, only minimal guidelines can be given. This step is, however, essential and should not be combined with Step 3, since the first derivation of an argument by even an experienced proof writer is seldom the best presentation of that argument.

In the preliminary construction of a proof:

- Derive all assertions of existence.

Comment: For example, suppose you wish to prove $(\forall x \in U_x)(\exists y \in U_y)\, P(x, y)$. We must exhibit a specific y, likely expressed in terms of arbitrary x such that $P(x, y)$ is true. Typically, in the derivation, we assume $P(x, y)$ and work backwards to find y. (But this backwards derivation is inappropriate for the presentation.)

- Derive all arguments that connect some assumption P to a conclusion Q.

Comment: If there is a standard process (for example, a definition) for proving Q, follow that process and use assumption P when it is needed. Otherwise, try both starting with P and working forward to Q and starting with Q and working backwards to P. (In the presentation, however, you must proceed from P to Q.)

Check your proof against the given list of guidelines.

As you become more familiar with what is expected of a proof, this step will become automatic and will take place as you write your presentation. To insure that you gain familiarity with the guidelines we will, for now, include this review as a separate step.

GUIDELINES

1. The beginning of the proof is clearly marked with the word Proof.

2. The appropriate form is followed.

3. The reader is informed of special proof techniques (contrapositive, contradiction, induction) to be employed.

4. All variables that appear in the proof are introduced in the appropriate context within the proof.

Recall that a variable has been properly introduced if it has been assigned a specific value from its universal set or it has been quantified and its universal set indicated.

5. The proof is written in complete English sentences.

In written proofs, do not use shorthand symbols for English words or phrases. For instance, symbols such as $\wedge$, $\vee$, $\rightarrow$, $\sim$, $\exists$, $\forall$, and $|$ (for divides) should not appear in your written proofs. (Use them all you want in the construction.) Also, symbolic representations of statements (such as $P(x)$, $Q(x, y)$) are inappropriate in the presentation.

Do not use mathematical symbols as shorthand for an English word or phrase in a written statement. For example, "the integer = the sum of two primes" is a misuse of "=". We may either write "the integer is the sum of two primes" or introduce variables and write "$n = p + q$ where p and q are prime."

This does not preclude using mathematical symbols in a proof, but they should appear in the context of a complete sentence (see the examples below.)

Finally, a complete sentence usually begins with a capital letter, somewhere includes a verb (which may be a mathematical symbol), and ends with a period.

6. All assumptions used in the proof are stated in the proof.

In particular, do not rely upon the statement that is to be proved to convey to your reader the assumptions you are making in the proof.

7. All logical arguments begin with the appropriate assumption, end with the desired conclusion and provide sufficient detail to lead the reader through the argument.

Example 2: (a) Review the proof given below against guidelines 1 - 7.

(b) Present a proof that follows the guidelines.

To Prove: If m and n are integers such that $m + n$ is even, then $m - n$ is also even.

Proof:
$m + n = 2k$
$m - n = 2k - 2n = 2(k - n)$

(a) **Review of the Proof**:

Guideline 2: The proof does not follow the appropriate form. Indeed, adherence to the appropriate form would solve many of the problems with the given presentation.

The statement to be proved has the form $(\forall m, n \in \mathbf{Z})(m + n \text{ even } \rightarrow m - n \text{ even})$. Therefore, following Argument Form 6, the form of the proof should be:

- Let m, n be arbitrary (variable) integers.
- If useful, expand on what it means for m and n to be integers. (In this case, it is not useful.)
- Assume $m + n$ is even.
- If useful expand on the assumption that $m + n$ is even. (In this case we will apply the definition an even integer.)
- Give a logical argument that ends with the conclusion that $m - n$ is even.

Guideline 4: In the given presentation none of the variables are introduced. (**Note**: Following the appropriate form would have caused m and n to be properly introduced, but k also needs to be introduced.)

Guideline 5: The two equations do qualify as sentences but the reader needs assistance to interpret the significance of each equation.

Guideline 6: The writer assumes that $m+n$ is even but that assumption is not stated in the presentation.

Guideline 7: The argument begins with the appropriate (but unstated) assumption that $m+n$ is even and concludes correctly that $m-n$ is even, but the reader is not lead through the process of getting from the hypothesis to the conclusion.

(b) **Proof**: Let m and n be integers. Suppose $m+n$ is even. Then there exists an integer k such that $m+n=2k$. Therefore, $m-n=(m+n)-2n=2k-2n=2(k-n)$, so $m-n$ is even.

Exercise 3: In this exercise we will be examining the given proofs of the following statement:

To be Proved: For every positive real number x, if $x \neq 3$ then $x-6 > -\frac{9}{x}$.

Proof 1: $x-6 > -\frac{9}{x}$
$x^2-6x > -9$
$x^2-6x+9 > 0$
$(x-3)^2 > 0$ which is true.

Proof 2: $(x-3)^2 > 0$
$x^2-6x+9 > 0$
$x^2-6x > -9$
$x-6 > -\frac{9}{x}$

(a) Represent the given statement symbolically and outline the form of a direct proof.
(b) Critique Proof 1 above using guidelines 1 - 7.
(c) Critique Proof 2 above using guidelines 1 - 7.
(d) Present a proof that follows guidelines 1 - 7.

SECTION 2.5: EXERCISES

2.5.1. Consider the following statement:

Given Statement: For every integer n, if n is even then there exists an integer m such that m is a perfect square and $4(m + n - 1) = n^2$.

(a) Complete Step 1 of the four step procedure; that is, give a symbolic representation of the statement to be proved and outline the form of the proof.

(b) Complete Step 2 of the four step procedure; that is, give a preliminary construction of the proof. In particular, derive m, expressed in terms of n.

(c) Present a proof of the given statement.

(d) Critique your proof against guidelines 1 - 7.

2.5.2. The object of this exercise is to prove, using the definition, that the sequence of real numbers $\{n + 1/n\}_{n=1}^{\infty}$ is not a Cauchy sequence.

(a) Use Exercise 1.4.6 to complete the statement:

The sequence $\{n + 1/n\}_{n=1}^{\infty}$ is not a Cauchy sequence provided ....

Now represent the statement symbolically and outline the form of the proof.

(b) Give a preliminary construction of the proof. In particular, select a specific ϵ and for an arbitrary M choose m and n. [HINT: Show that for every integer $n \geq 1$ we have $(n + 1 + \frac{1}{n+1}) - (n + \frac{1}{n}) = 1 - \frac{1}{n(n+1)}$ and note that $1 - \frac{1}{n(n+1)} \geq \frac{1}{2}$.]

(c) Give a proof that the sequence of real numbers $\{n + 1/n\}_{n=1}^{\infty}$ is not a Cauchy sequence.

(d) Critique your proof using guidelines 1 - 7 and rewrite as necessary. When you believe that your proof follows all guidelines, state that that is the case.

2.5.3. The object of this exercise is to prove, using the definition, that $\lim_{x\to 0}(x^2 + 1) \neq 0$.

(a) Use Exercise 1.4.7 to complete the statement:

$\lim_{x\to 0}(x^2 + 1) \neq 0$ provided ....

Now represent the statement symbolically and outline the form of the proof.

(b) Give a preliminary construction of the proof. In particular, select a specific ϵ.

(c) Give a proof that $\lim_{x\to 0}(x^2 + 1) \neq 0$.

(d) Critique your proof using guidelines 1 - 7 and rewrite as necessary. When you believe that your proof follows all guidelines, state that that is the case.

2.5.4. The object of this exercise is to prove, using the definition, that $\lim_{x\to 3}(2x-4)=2$.

(a) Use Exercise 1.4.7 to complete the statement:
$\lim_{x\to 3}(2x-4)=2$ provided ....
Now represent the statement symbolically and outline the form of the proof.

(b) Give a preliminary construction of the proof. In particular, for an arbitrary ϵ derive a specific δ (expressed in terms of ϵ). [HINT: Work backwards from $|(2x-4)-2|<\epsilon$.]

(c) Give a proof that $\lim_{x\to 3}(2x-4)=2$.

(d) Critique your proof using guidelines 1 - 7 and rewrite as necessary. When you believe that your proof follows all guidelines, state that that is the case.