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AN INTRODUCTION TO MATHEMATICAL PROOFS

NOTES FOR MATH 3034

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NOTATION

Symbol	Set
\mathbf{R}	The set of all real numbers
\mathbf{R}^+	The set of all positive real numbers
$\mathbf{R}^\#$	The set of all nonzero real numbers
\mathbf{Q}	The set of all rational numbers
\mathbf{Q}^+	The set of all positive rational numbers
$\mathbf{Q}^\#$	The set of all nonzero rational numbers
\mathbf{Z}	The set of all integers
\mathbf{Z}^+	The set of all positive integers
$\mathbf{Z}^\#$	The set of all nonzero integers
$\mathbf{M}_{m \times n}(\mathbf{S})$	The set of all $m \times n$ matrices with entries from the set S .

CHAPTER 1: THE STRUCTURE OF MATHEMATICAL STATEMENTS

Section 1.1: STATEMENTS

Definition 1: A **statement** (or proposition) is an assertion that is true or false but not both.

Example 1: Which of the following is a statement:

1. 4 is a prime integer.
2. $2^{400} + 1$ is a prime integer.
3. Abe Lincoln was a great president.
4. $2^{400} + 1$ is a large number.
5. $x \leq 5$.

Solution:

- (1) Yes (2) Yes (3) No (4) No
(5) This is not a statement until x is assigned a particular value. This is an example of an **open statement** and we will consider these later. In the meantime, we will use open statements in examples as if they were statements.

Definition 2: Let P denote a statement. The **negation** of P , denoted by $\sim P$, is the assertion that P is false.

The following truth table relates the truth values of P and $\sim P$.

Mathematical Term	English Expression	Symbolic Form	Truth Table Definition						
Negation	not P	$\sim P$	<table><tr><td>P</td><td>$\sim P$</td></tr><tr><td>T</td><td>F</td></tr><tr><td>F</td><td>T</td></tr></table>	P	$\sim P$	T	F	F	T
P	$\sim P$								
T	F								
F	T								

Exercise 1: In each of the following, write $\sim P$ in English.

- (a) P : $\pi < 4$.
- (b) P : $\ln(e^2 x^2) = 2(1 + \ln(x))$
- (c) P : 9 is an even integer.

We now introduce “**connectives**” that can be used to combine two or more given statements to obtain a new statement. Let P and Q be statements.

Conjunction and Disjunction (“and” and “or”)

Mathematical Term	English Expression	Symbolic Form	Truth Table Definition		
			P	Q	$P \wedge Q$
conjunction	P and Q	$P \wedge Q$	T	T	T
			T	F	F
			F	T	F
			F	F	F
disjunction	P or Q	$P \vee Q$	P	Q	$P \vee Q$
			T	T	T
			T	F	T
			F	T	T
			F	F	F

Comments

- (a) Other terms that mean “and” are **but** and **also**.
- (b) The mathematical “or” is always used in the “inclusive” sense; that is, if both P and Q are true then $P \vee Q$ is also true.

In common English usage, “or” is sometimes used in an “exclusive” sense; that is, one or the other, but not both, is true. A statement such as “You may have coke or pepsi,” is likely meant in the exclusive sense.

Exercise 2: With statements P and Q as defined below, write each of the statements in (a) – (f) symbolically and state whether the statement is true or false.

P : 4 is an even integer. Q : $3 < \sqrt{17}$

- (a) 4 is an even integer and $3 < \sqrt{17}$.
- (b) 4 is an even integer and $3 \geq \sqrt{17}$.
- (c) 4 is an odd integer and $3 \geq \sqrt{17}$.
- (d) 4 is an even integer or $3 < \sqrt{17}$.
- (e) 4 is an odd integer or $3 < \sqrt{17}$.
- (f) 4 is an odd integer or $3 \geq \sqrt{17}$.

Exercise 3: Write the statement “19 is neither even nor divisible by 3” in symbolic form using the notation:

P : 19 is even Q : 19 is divisible by 3

Implication

Mathematical Term	English Expression	Symbolic Form	Truth Table Definition
implication	If P then Q	$P \rightarrow Q$	P Q $P \rightarrow Q$
			T T T
			T F F
			F T T
			F F T

Comments

(a) Note that the implication $P \rightarrow Q$ is true whenever P is false. For example the statement “If 4 is an odd integer then 18 is prime,” is a true statement.

(b) Following are some of the various English expressions that translate to $P \rightarrow Q$.

- If P then Q .
- P implies Q .
- P only if Q .
- Q when P .
- Q is necessary for P .
- P is sufficient for Q .

One way to remember these is to remember that in an implication, certain key words always refer to the hypothesis and others to the conclusion. The following table summarizes some of these.

Terms for P and Q in the implication $P \rightarrow Q$

P	Q
hypothesis	conclusion
if	only if
sufficient	necessary

Exercise 4: In each of the following you are given statements P and Q and an implication involving P and Q . In each problem determine if the given statement is $P \rightarrow Q$ or $Q \rightarrow P$. Also determine if the given statement is true or false.

(a) $5 = 7$ whenever $\sqrt{3} \neq 2$.

P : $5 = 7$ Q : $\sqrt{3} \neq 2$.

(b) 3 is even only if 6 is prime.

P : 3 is even Q : 6 is prime

(c) For 10 to be prime it is necessary that 4 be even.

P : 10 is prime Q : 4 is even

(d) For 10 to be prime it is sufficient that 4 be even.

P : 10 is prime Q : 4 is even

Equivalence

Mathematical Term	English Expression	Symbolic Form	Truth Table Definition
equivalence	P if and only if Q	$P \leftrightarrow Q$	P Q $P \leftrightarrow Q$
			T T T
			T F F
			F T F
			F F T

Following are some of the various English expressions that translate to $P \leftrightarrow Q$.

- (1) P if and only if Q .
- (2) P is equivalent to Q .
- (3) P is necessary and sufficient for Q .

Exercise 5: In each of (a) – (c), determine if the statement is true or false.

(a) 4 is an even integer if and only if $3 < \sqrt{17}$.

(b) 4 is an odd integer if and only if $3 < \sqrt{17}$.

(c) 4 is an odd integer if and only if $3 \geq \sqrt{17}$.

The Converse and Contrapositive

An implication $P \rightarrow Q$ has **converse** $Q \rightarrow P$ and **contrapositive** $\sim Q \rightarrow \sim P$.

Exercise 6: For each statement in (a) – (d) write both the converse and contrapositive of the given statement. In each case, determine whether the given statement is true or false, state whether the converse is true or false, and state whether the contrapositive is true or false.

- (a) If $3 < \sqrt{17}$ then 4 is not a prime integer.
- (b) If $3 \geq \sqrt{17}$ then 4 is not a prime integer.
- (c) If $3 < \sqrt{17}$ then 4 is a prime integer.
- (d) If $3 \geq \sqrt{17}$ then 4 is is a prime integer.

1.1 EXERCISES

1.1.1. Which of the following is a statement?

- (a) $2^{300} > 3^{200}$.
- (b) The solutions to $x^3 - 3x^2 + 4x - 6 = 0$ are difficult to find.
- (c) $4198 + 7432$.
- (d) $853 = (56)15 + 13$.
- (e) There is a prime integer larger than $10^{10^{10}}$.

1.1.2. In each of (a) – (i) determine the value(s) of a and/or b for which the statement is true.

- (a) $3 < 2$ and $b = 6$.
- (b) $a = 4$ or $2 < 3$.
- (c) $a = 4$ or $3 < 2$.
- (d) If $a = 4$ then $2 < 3$.
- (e) If $a = 4$ then $3 < 2$.
- (f) If $2 < 3$ then $b = 6$.
- (g) If $3 < 2$ then $b = 6$.
- (h) $a = 4$ if and only if $2 < 3$.
- (i) $3 < 2$ if and only if $b = 6$.

1.1.3. Identify the hypothesis and conclusion in each of the following implications.

- (a) 4 is an even integer only if 3 is prime.
- (b) For 4 to be even it is sufficient that 3 be prime.
- (c) For 3 to be prime it is necessary that 4 be even.
- (d) For 4 to be even, 3 must be prime.
- (e) 4 is even when 3 is prime.
- (f) 3 is prime if 4 is even.

1.1.4. Write the converse and contrapositive of each of the following.

- (a) If $3 \leq \sqrt{17}$ then $\sqrt{7} > 2.5$.
- (b) If $3 \leq \sqrt{17}$ then $\sqrt{7} \neq 2.5$.
- (c) If $5 > \sqrt{17}$ then $\sqrt{7} = 2.5$.

1.1.5. Suppose the following statements are all true:

- If Joe passes Math 3034 then Joe will graduate.
- Either Joe will pass Math 3034 or he will get a job flipping hamburgers.
- Joe will not graduate.

Determine whether or not Joe will get a job flipping hamburgers. Explain your reasoning.

1.1.6. Knights always tell the truth but knaves never tell the truth. In a group of three individuals (who we will label as #1, #2, and #3) each is either a knight or a knave. Each makes a statement:

#1: “We are all three knaves.”

#2: “Two of us are knaves and one of us is a knight.”

#3: “I am a knight and the other two are knaves.”

Which are knights and which are knaves? Explain your reasoning.

1.1.7. The prom is Saturday night and the following facts are known:

- Either Mike is not taking Jen or Jason is going with Debbie.
- If Jason goes with Debbie and Bill stays home then Sue will go with Robbie.
- Bill is staying home but Sue will not go with Robbie.

Is Mike taking Jen to the prom? Justify your answer.

Section 1.2. STATEMENT FORMS, LOGICAL EQUIVALENCE, AND NEGATIONS

Logical Equivalence

An expression such as $(P \wedge \sim Q) \rightarrow \sim (R \vee S)$, where P , Q , R , and S represent unassigned statements, is called a **statement form**. While a statement must be either true or false, a statement form has no meaning and no truth value until each variable is replaced by a specific statement.

Definition 1: Two statement forms are **logically equivalent** provided they have the same truth values for all possible truth values of the component variables.

Notation: If R and S are statement forms, we will write $R \iff S$ to mean that R and S are logically equivalent.

One method for determining whether two statement forms are logically equivalent is to construct a **truth table** that compares their truth values for all possible truth values of the component variables. The proofs of the following theorems will illustrate.

Theorem 1: An implication and its contrapositive are logically equivalent; that is, for components P and Q ,

$$(P \rightarrow Q) \iff (\sim Q \rightarrow \sim P).$$

Proof:

P	Q	$\sim P$	$\sim Q$	$P \rightarrow Q$	$\sim Q \rightarrow \sim P$
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

Theorem 2: An implication and its converse are not logically equivalent; that is, for components P and Q ,

$$(P \rightarrow Q) \not\iff (Q \rightarrow P).$$

Proof:

P	Q	$P \rightarrow Q$	$Q \rightarrow P$
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	T

Note that in the second and third rows the implication and its converse have different truth values.

Theorem 3: For symbolic statements P and Q ,

$$(P \rightarrow Q) \iff (\sim P \vee Q).$$

Proof:

P	Q	$\sim P$	$P \rightarrow Q$	$\sim P \vee Q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Comment: Equivalence (represented by \leftrightarrow) and logical equivalence (represented by \iff) are not the same. Recall that particular statements are equivalent provided both are true or both are false. Only statement forms, not statements, are logically equivalent.

The importance of logically equivalent forms is that when specific component statements are “plugged in,” the resulting statements are always equivalent. For instance, since the statement forms $P \rightarrow Q$ and $\sim Q \rightarrow \sim P$ are logically equivalent, it automatically follows that the statements:

“If $3 < \sqrt{17}$ then 5 is prime”, and

“If 5 is not prime then $3 \geq \sqrt{17}$ ”

are equivalent.

Rules for Negations

In mathematics we must routinely formulate and understand the negation of a given statement. In one sense, the negation of a statement is trivial to formulate. For example we can merely say, “it is not true that” For statement forms we can merely write the negation of $[P \rightarrow (Q \vee R)]$ as $\sim [P \rightarrow (Q \vee R)]$.

Formulations such as those above are not particularly useful. By a **useful negation** we mean that in the corresponding form, the \sim has been distributed across all connectives and

there are no double negatives. The following theorem provides the basic rules for writing useful negations.

Theorem 4: Let P and Q be statement forms.

- (a) $\sim(\sim P) \iff P$.
- (b) $\sim(P \wedge Q) \iff (\sim P \vee \sim Q)$.
- (c) $\sim(P \vee Q) \iff (\sim P \wedge \sim Q)$.
- (d) $\sim(P \rightarrow Q) \iff (P \wedge \sim Q)$.

Proof of (b):

P	Q	$\sim P$	$\sim Q$	$P \wedge Q$	$\sim(P \wedge Q)$	$\sim P \vee \sim Q$
T	T	F	F	T	F	F
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T

Proof of (d): Using Theorem 3, and parts (a) and (c) of Theorem 4, note that

$$\sim(P \rightarrow Q) \iff \sim(\sim P \vee Q) \iff (\sim(\sim P) \wedge \sim Q) \iff (P \wedge \sim Q).$$

Example 1: Give a useful and logically equivalent formulation for the statement form $\sim[P \rightarrow (\sim Q \rightarrow R)]$.

Solution:

$$\sim[P \rightarrow (\sim Q \rightarrow R)] \iff [P \wedge \sim(\sim Q \rightarrow R)] \iff [P \wedge (\sim Q \wedge \sim R)].$$

Exercise 1: Give a useful and logically equivalent form of each of the following.

- a. $\sim[(P \wedge \sim Q) \rightarrow R]$
- b. $\sim[P \rightarrow (Q \vee \sim R)]$
- c. $\sim[P \wedge (Q \rightarrow R)]$

Exercise 2: In each of (a) – (d):

- (i) Write the statement in symbolic form. (This requires assigning labels to the components.)
 - (ii) Write the negation in useful symbolic form.
 - (iii) Write the negation in useful form as an English statement.
 - (iv) Which is true, the statement or its negation.
-
- (a) 2 is even but 2 is also prime.
 - (b) Either 5 is odd or 5 is a multiple of 3.
 - (c) 9 is neither even nor prime.
 - (d) For 8 to be odd but not prime, it is sufficient that 8 be a multiple of 4.

More Equivalencies

Terminology: A statement form T that is true for all truth values of its components is called a **tautology**. A statement form C that is false for all truth values of its components is called a **contradiction**.

Example 2: If P is a statement form then $P \vee \sim P$ is a tautology and $P \wedge \sim P$ is a contradiction.

Theorem 5 (Basic Equivalencies): Let P, Q, R, T , and C be statement forms, where T is a tautology and C is a contradiction.

- 1. Commutative Laws
 - (a) $P \wedge Q \iff Q \wedge P$
 - (b) $P \vee Q \iff Q \vee P$
- 2. Associative Laws
 - (a) $(P \wedge Q) \wedge R \iff P \wedge (Q \wedge R)$
 - (b) $(P \vee Q) \vee R \iff P \vee (Q \vee R)$
- 3. Distributive Laws
 - (a) $P \wedge (Q \vee R) \iff (P \wedge Q) \vee (P \wedge R)$
 - (b) $P \vee (Q \wedge R) \iff (P \vee Q) \wedge (P \vee R)$
- 4. Idempotent Laws
 - (a) $P \wedge P \iff P$
 - (b) $P \vee P \iff P$
- 5. Identity Laws
 - (a) $P \wedge T \iff P$
 - (b) $P \vee C \iff P$

6. Universal Bound Laws (a) $P \vee T \iff T$
 (b) $P \wedge C \iff C$
7. Absorption Laws (a) $P \vee (P \wedge Q) \iff P$
 (b) $P \wedge (P \vee Q) \iff P$

Proof of 3(a): We will prove 3(a) with a truth table. Note that there are three components in the statement form and for each component there are two possible truth values – true or false. Thus, the total number of possibilities (that is, the number of lines in the truth table) is $2^3 = 8$.

P	Q	R	$Q \vee R$	$P \wedge (Q \vee R)$	$P \wedge Q$	$P \wedge R$	$(P \wedge Q) \vee (P \wedge R)$
T	T	T	T	T	T	T	T
T	T	F	T	T	T	F	T
T	F	T	T	T	F	T	T
T	F	F	F	F	F	F	F
F	T	T	T	F	F	F	F
F	T	F	T	F	F	F	F
F	F	T	T	F	F	F	F
F	F	F	F	F	F	F	F

Note that columns 5 and 8 of the table have the same truth values, thus proving 3(a).

Example 3: Use the basic equivalences of Theorems 3 and 5 to prove that the statement forms $P \rightarrow (Q \vee R)$ and $(P \wedge \sim Q) \rightarrow R$ are logically equivalent.

Proof: $P \rightarrow (Q \vee R) \iff \sim P \vee (Q \vee R) \iff (\sim P \vee Q) \vee R \iff \sim(\sim P \vee Q) \rightarrow R \iff (P \wedge \sim Q) \rightarrow R$.

Comment: We will later find the equivalence given in Example 3 to be quite useful. To prove a statement of the form $P \rightarrow (Q \vee R)$ involves assuming P and trying to conclude the awkward form $(Q \vee R)$. If we substitute the equivalent form, $(P \wedge \sim Q) \rightarrow R$, we begin with two hypotheses, P and $\sim Q$, and need only to prove the single conclusion R .

SECTION 1.2: EXERCISES

1.2.1. Use a truth table to prove Theorem 4(c).

1.2.2. Use truth tables to determine whether or not each of the following pairs of statement forms are logically equivalent.

- (a) $\sim (P \rightarrow Q)$ and $\sim P \rightarrow \sim Q$.
- (b) $(P \rightarrow Q) \rightarrow R$ and $(P \rightarrow R) \rightarrow Q$.

1.2.3. In (a) – (d) do each of the following steps:

- (i) Use the assigned variables and the symbols \sim , \wedge , and \vee to write the statement in symbolic form;
- (ii) Write a useful (simplified) negation of the statement in symbolic form; and
- (iii) Write a useful negation of the statement in English.

(a) 9 is an odd integer but is not a prime.

P : 9 is an odd integer Q : 9 is a prime.

(b) 9 is neither an odd integer nor a prime. (Same symbols as (a).)

(c) Mike is neither healthy nor wealthy, but Mike is wise.

P : Mike is healthy Q : Mike is wealthy R : Mike is wise

(d) Mike is healthy, wealthy, and wise. (Same symbols as (c).)

1.2.4. Give the converse, the contrapositive, and the negation (in useful form) of each of the following statement forms.

(a) $(P \vee Q) \rightarrow R$

(b) $(P \wedge Q) \rightarrow R$

(c) $P \rightarrow (\sim Q \wedge R)$

(d) $(P \rightarrow Q) \rightarrow R$

(e) $P \rightarrow (Q \rightarrow R)$

1.2.5. Write a useful negation of each of the following statements.

(a) If 3 is even and 7 is prime then $3 > 2$.

(b) If 4 is even then 5 is neither odd nor prime.

(c) If $|x - 4| < 2$ then $-2 < x < 2$.

(d) If $|x - 4| > 2$ then either $x < -2$ or $x > 2$.

1.2.6. Use a truth table to prove Theorem 5, part 3(b).

1.2.7. Use the Theorems of Section 1.2 to prove the following equivalencies.

(a) $[(P \vee Q) \rightarrow R] \iff [(P \rightarrow R) \wedge (Q \rightarrow R)]$

(b) $[P \rightarrow (Q \rightarrow R)] \iff [(P \wedge Q) \rightarrow R]$

(c) $[(P \rightarrow Q) \rightarrow R] \iff [(\sim P \rightarrow R) \wedge (Q \rightarrow R)]$

1.2.8. The famous detective, Hercule Poirot, has arrived at the following facts.

- The statement, “if Col. Mustard did not commit the murder then neither Miss Scarlet nor Mr. Green committed the murder” is false.
- Either Miss Scarlet did not commit the murder or the weapon was a candlestick.
- If the weapon was a candlestick then Col. Mustard committed the murder.

Who committed the murder? Explain how you know.

Section 1.3: QUANTIFIERS

Consider the assertions:

$P(n)$: n is a prime integer.

$Q(x, y)$: $2x + y = 3$.

In each case the assertion may be either true or false depending upon the value assigned to the variables. $P(n)$ and $Q(x, y)$ are examples of **open statements**; that is, assertions that contain variables. When appropriate values are assigned to the variables in an open statement, it becomes a statement. For example:

- $P(4)$ is a false statement.
- $P(5)$ is a true statement.
- $P(2^{23} + 1)$ is a statement, but we cannot tell whether it is true or false.
- $Q(1, 1)$ is a true statement.
- $Q(1, 2)$ is a false statement.

Let $P(x)$ denote an open statement. Associated with the variable x is a **universal set** (or domain) U_x such that for each $a \in U_x$, $P(a)$ is a statement; that is, $P(a)$ is either true or false.

The **truth set** of $P(x)$ is the set of values in U_x for which $P(x)$ becomes a true statement.

Example 1: Let $P(x)$ denote the statement: $x^2 - 4 = 0$. Let $U_x = \mathbf{R}$, where \mathbf{R} denotes the set of all real numbers. The truth set of $P(x)$ is the set $\{-2, 2\}$.

Definition 1: (The Existential Quantifier) Let $P(x)$ be an open statement. The statement

“There exists x in U_x such that $P(x)$,”

written symbolically as $(\exists x \in U_x) P(x)$, is defined to be true provided the truth set of $P(x)$ is not the empty set.

Example 2: Consider the statement:

“There exist real numbers x and y such that $2x + y^2 = 6$.”

Write the statement in symbolic form and verify that the statement is true.

Solution: Let $Q(x, y)$ denote the open statement $2x + y^2 = 6$. Note that the universal sets for x and y are given to be the set \mathbf{R} of real numbers. Thus, the given statement has

symbolic form $(\exists x, y \in \mathbf{R}) Q(x, y)$. Since $Q(3, 0)$ is a true statement, the given statement is also true.

Exercise 1: Consider the statement:

“There exists a prime integer p such that $p > 10$ and p is even.”

Use the following assignments to write the given statement in symbolic form. Is the statement true or false?

U_p : all prime integers

$Q(p)$: $p > 10$

$R(p)$: p is even

Definition 2: (The **Universal Quantifier**) Let $P(x)$ be an open statement. The statement

“For all x in U_x , $P(x)$,”

written symbolically as $(\forall x \in U_x) P(x)$, is defined to be true provided U_x is the truth set of $P(x)$.

Comment: A statement of the form

“If $x \in U_x$ then $P(x)$ ”

translates to

“For all x in U_x , $P(x)$ ”

Example 3: Consider the following statements:

(a) There exist real numbers x and y such that $x^2 + y^2 < 0$.

(b) There exist real numbers x and y such that $x^2 + y^2 = 0$.

(c) If x and y are real numbers then $x^2 + y^2 = 0$.

(d) If x and y are real numbers then $x^2 + y^2 \geq 0$.

Use the assignment of symbols:

$U_x = U_y = \mathbf{R}$ $P(x, y)$: $x^2 + y^2 < 0$

$Q(x, y)$: $x^2 + y^2 = 0$ $R(x, y)$: $x^2 + y^2 \geq 0$

Write each statement symbolically and state whether it is true or false.

Solution: (a) $(\exists x, y \in \mathbf{R}) P(x, y)$. The statement is false.

(b) $(\exists x, y \in \mathbf{R}) Q(x, y)$. The statement is true (take $x = y = 0$).

(c) $(\forall x, y \in \mathbf{R}) Q(x, y)$. The statement is false. (For instance, $Q(1, 1)$ is false.)

(d) $(\forall x, y \in \mathbf{R}) R(x, y)$. The statement is true.

Exercise 2: In each of the following open statements, the universal sets are $U_x = U_y = \mathbf{R}$, where \mathbf{R} denotes set of all real numbers.

$$P(x): x^2 - 5x + 6 = 0 \quad Q(x): \sin^2 x + \cos^2 x = 1$$

$$R(x): x^2 < 0 \quad S(x, y): x + y = 10$$

In each of (a) – (j) below:

- (i) write the statement as an English sentence that communicates the universal set for each variable; and
- (ii) determine if the statement is true or false.

- a. $(\forall x \in \mathbf{R}) P(x)$
- b. $(\exists x \in \mathbf{R}), P(x)$
- c. $(\forall x \in \mathbf{R}) Q(x)$
- d. $(\exists x \in \mathbf{R}), Q(x)$
- e. $(\forall x \in \mathbf{R}) R(x)$
- f. $(\exists x \in \mathbf{R}), R(x)$
- g. $(\forall x, y \in \mathbf{R}) S(x, y)$
- h. $(\exists x, y \in \mathbf{R}) S(x, y)$
- i. $(\forall x \in \mathbf{R}) (\exists y \in \mathbf{R}), S(x, y)$
- j. $(\exists y \in \mathbf{R}) (\forall x \in \mathbf{R}), S(x, y)$

Comment: (cf (i) and (j) in Exercise 2 above) Note that

$$(\forall x \in U_x) (\exists y \in U_y) S(x, y) \not\iff (\exists y \in U_y) (\forall x \in U_x) S(x, y).$$

Comment: An open statement becomes a statement when each variable appearing is either assigned a specific value from the universal set or is quantified.

In a mathematical statement that contains variables, each variable must be **properly introduced**; that is, each variable must either

- be assigned a specific value, or
- be quantified and have its universal set indicated in the statement.

Theorem 1 (Rules for Negations): Let $P(x)$ denote an open statement. Then

- (a) $\sim [(\exists x \in U_x) P(x)] \iff (\forall x \in U_x) \sim P(x).$
- (b) $\sim [(\forall x \in U_x) P(x)] \iff (\exists x \in U_x) \sim P(x).$

Example 4: Give a useful logically equivalent form for the negation of:

$$(\forall x) [(\exists y) P(x, y) \vee (\forall z) Q(x, z)]$$

Solution: $(\exists x) [(\forall y) \sim P(x, y) \wedge (\exists z) \sim Q(x, z)]$.

Exercise 3: Give a useful, logically equivalent form for the negation of each of the following:

(a) $(\forall x) [(\exists y) P(x, y) \wedge (\forall z) (\exists w) Q(x, z, w)]$

(b) $(\forall x) [(\exists y) P(x, y) \rightarrow (\forall z) (Q(x, z) \wedge (\exists w) \sim R(x, w, z))]$

Exercise 4: In each of (a) and (b) below:

(i) Assign a universal set to each variable, label each component statement with a symbol, and write the given statement symbolically;

(ii) write a useful negation of the statement symbolically; and

(iii) Give a useful written negation of the statement.

(a) For every real number y there is a positive real number x such that $y = \ln x$.

(b) For every positive real number y there exist real numbers x and z such that $y = x^2 - 1$ and $y = e^z$.

Example 5: Consider the statement form:

$$(\forall x \in U_x) [(\exists y \in U_y) P(x, y) \rightarrow (\forall z \in U_z) Q(x, z)].$$

The **contrapositive** is:

$$(\forall x \in U_x) [(\exists z \in U_z) \sim Q(x, z) \rightarrow (\forall y \in U_y) \sim P(x, y)].$$

The **converse** is:

$$(\forall x \in U_x) [(\forall z \in U_z) Q(x, z) \rightarrow (\exists y \in U_y) P(x, y)].$$

The **negation** is:

$$(\exists x \in U_x) [(\exists y \in U_y) P(x, y) \wedge (\exists z \in U_z) \sim Q(x, z)].$$

Example 6: Consider the statement:

For all real numbers x , if $x^2 - 5x + 6 = 0$ then either $x = 2$ or $x = 3$.

- (i) Assign a universal set to each variable, label each component statement with a symbol, and write the given statement symbolically;
- (ii) write a useful negation of the statement symbolically;
- (iii) give a useful written negation of the statement;
- (iv) write the contrapositive of the implication symbolically; and
- (v) write the contrapositive as an English sentence.

Solution: (i) $U_x = \mathbf{R}$ $P(x): x^2 - 5x + 6 = 0$ $Q(x): x = 2$

$R(x): x = 3$.

Symbolic Statement: $(\forall x \in \mathbf{R}) [P(x) \rightarrow (Q(x) \vee R(x))]$

(ii) Symbolic Negation: $(\exists x \in \mathbf{R}) [P(x) \wedge \sim Q(x) \wedge \sim R(x)]$

(iii) Written Negation: There exists a real number x such that $x^2 - 5x + 6 = 0$ but $x \neq 2$ and $x \neq 3$.

(iv) Symbolic Contrapositive: $(\forall x \in \mathbf{R}) [(\sim Q(x) \wedge \sim R(x)) \rightarrow \sim P(x)]$.

(v) Written Contrapositive: For all real numbers x , if $x \neq 2$ and $x \neq 3$ then $x^2 - 5x + 6 \neq 0$.

Exercise 5: In each of (a) and (b) below:

- (i) Assign a universal set to each variable, label each component statement with a symbol, and write the given statement symbolically;
 - (ii) write a useful negation of the statement symbolically;
 - (iii) give a useful written negation of the statement;
 - (iv) write the contrapositive of the implication symbolically; and
 - (v) write the contrapositive as an English sentence.
- (a) For integers m and n , if $m + n$ is even, then both m and n are even.
- (b) For every positive integer n , if n is not prime then there exists a prime integer p such that $p \leq \sqrt{n}$ and p is a factor of n .

SECTION 1.3: EXERCISES

1.3.1. In each of (a) – (c), give a useful negation of the statement form.

- (a) $(\forall x \in U_x) [(\exists y \in U_y) P(x, y) \vee (\forall z \in U_z) Q(x, z)]$.
- (b) $(\exists x \in U_x) \left[(\forall y \in U_y) P(x, y) \rightarrow \left((\forall z \in U_z) Q(x, z) \vee (\exists w \in U_w) R(x, w) \right) \right]$.
- (c) $(\exists x \in U_x) \left[(\forall y \in U_y) P(x, y) \rightarrow (\forall z \in U_z) (\exists w \in U_w) \left(Q(x, z, w) \wedge \sim R(x, z, w) \right) \right]$.

In Exercises 1.3.2 – 1.3.5: Do each of the following steps:

- (i) Label all the components (e.g. $P(x)$: $x > 0$; $Q(x)$: $x^2 = 1$.) then give a symbolic representation of the statement using the given universal set.
- (ii) Give the symbolic form of a useful negation of the statement.
- (iii) Give a useful written negation of the statement.

1.3.2. **Statement:** For every 2×2 matrix A there exists a 2×2 matrix B such $AB \neq BA$.

$U_A = U_B = M_{2 \times 2}(\mathbf{R})$, the set of all 2×2 matrices with entries from \mathbf{R} .

1.3.3. **Statement:** There exist positive integers m and n such that $2m + 3n = 4$.

$U_m = U_n = \mathbf{Z}^+$, the set of all positive integers.

1.3.4. **Statement:** For every pair of real numbers a and b , either $a < b$ or $b < a$.

$U_a = U_b = \mathbf{R}$, the set of all real numbers.

1.3.5. **Statement:** For all real numbers a and b there exists a real number c such that $a + c = 2$ and $b + c = 3$.

$U_a = U_b = U_c = \mathbf{R}$, the set of all real numbers.

1.3.6. In each of (a) and (b), give the contrapositive of the given statement form.

- (a) $(\exists x \in U_x) \left[(\forall y \in U_y) P(x, y) \rightarrow \left((\forall z \in U_z) Q(x, z) \vee (\exists w \in U_w) R(x, w) \right) \right]$.
- (b) $(\exists x \in U_x) \left[(\forall y \in U_y) P(x, y) \rightarrow (\forall z \in U_z) (\exists w \in U_w) \left(Q(x, z, w) \wedge \sim R(x, z, w) \right) \right]$.

1.3.7. In each of (a) and (b):

- (i) give a useful written negation of the statement, and
- (ii) give a useful written contrapositive of the statement.
- (a) For all real numbers x and y , if $x < y$ then there exists a rational number r such that $x < r < y$.
- (b) For every real number x , if $0 < x < 1$ then there exists a positive real number ϵ such that $\epsilon < x < 1 - \epsilon$.

Section 1.4: DEFINITIONS

The general symbolic form of a **definition** is

$$(\forall x \in U_x) [N(x) \leftrightarrow D(x, y_1, y_2, \dots)]$$

where $N(x)$ introduces the term being defined and $D(x, y_1, y_2, \dots)$ gives the definition of the term.

Example 1: Express the following definition in symbolic form:

Definition: A real number r is a **rational number** provided there exist integers m and n such that $n \neq 0$ and $r = m/n$.

Solution:

Assignment of Symbols:

$U_r = \mathbf{R}$, the set of all real numbers.

$U_m = U_n = \mathbf{Z}$, the set of all integers.

$N(r)$: r is a rational number.

$P(n)$: $n \neq 0$.

$Q(r, m, n)$: $r = m/n$.

Symbolic Form of the Definition:

$$(\forall r \in \mathbf{R}) [N(r) \leftrightarrow (\exists m, n \in \mathbf{Z}) (P(n) \wedge Q(r, m, n))].$$

Thus, the definition has general form $(\forall r \in \mathbf{R}) [N(r) \leftrightarrow D(r, m, n)]$ indicated above, where $D(r, m, n)$ denotes the statement $(\exists m, n \in \mathbf{Z}) (P(n) \wedge Q(r, m, n))$.

We will not have occasion to negate an entire definition. Associated with every definition “ x is a term,” however, is the corresponding definition of “ x is **not** a term”. This definition has the general form

$$(\forall x \in U) [\sim N(x) \leftrightarrow \sim D(x, y_1, y_2, \dots)]$$

Example 2: Continuing from Example 1, give the symbolic form then the English statement of the definition of “ r is not a rational number.”

Solution: The symbolic form is:

$$(\forall r \in \mathbf{R}) [\sim N(r) \leftrightarrow (\forall m, n \in \mathbf{Z}) (\sim P(n) \vee \sim Q(r, m, n))].$$

The English statement is:

A real number r is a not rational number provided for all integers m and n either $n = 0$ or $r \neq m/n$.

Example 3:

Definition: A function f is **bounded on the real numbers** provided there exists a positive real number M such that for every real number x , $|f(x)| < M$.

For the definition above:

- (a) Write the definition in symbolic form.
- (b) Give the symbolic form of the definition of: f is not bounded on the real numbers.
- (c) Write out a useful definition of: f is not bounded on the real numbers.

Solution: (a) U_f is the set of all functions defined on the set of real numbers.

$U_M = \mathbf{R}^+$, the set of positive real numbers.

$U_x = \mathbf{R}$, the set of all real numbers.

$N(f)$: f is bounded on the real numbers.

$P(f, M, x)$: $|f(x)| < M$.

Symbolic Form: $(\forall f) [N(f) \leftrightarrow (\exists M \in \mathbf{R}^+) (\forall x \in \mathbf{R}) P(f, M, x)]$.

(b) $(\forall f) [\sim N(f) \leftrightarrow (\forall M \in \mathbf{R}^+) (\exists x \in \mathbf{R}) \sim P(f, M, x)]$.

(c) A function f is **not bounded on the real numbers** provided for every positive real number M there exists a real number x such that $|f(x)| \geq M$.

Exercise 1:

Definition: A real number u is a **least upper bound for a set A of real numbers** provided $a \leq u$ for every $a \in A$ and for every positive real number ϵ there exists $b \in A$ such that $u - \epsilon < b$.

(a) For the definition above use the assignment of symbols below to write the definition in symbolic form.

$U_u = \mathbf{R}$, where \mathbf{R} is the set of all real numbers.

U_A is the set of all subsets of \mathbf{R} .

$U_\epsilon = \mathbf{R}^+$, the set of positive real numbers.

$U_a = U_b = A$.

$N(u, A)$: u is a **least upper bound for a set A of real numbers**

$P(a, u)$: $a \leq u$.

$Q(b, \epsilon, u)$: $u - \epsilon < b$.

(b) Give the symbolic form of the definition of: u is not a least upper bound for a set A of real numbers.

(c) Write out a useful definition of: u is not a least upper bound for a set A of real numbers.

Some Definitions

We conclude this section with some definitions that will be useful in our later proofs.

Definition 1: An integer n is even provided there exists an integer k such that $n = 2k$.

Definition 2: An integer n is odd provided there exists an integer k such that $n = 2k + 1$.

Definition 3: The integer b divides the integer a provided there exists an integer c such that $a = bc$.

Comment: We will write $b|a$ to denote symbolically that b divides a .

Note that the definition of “divides” does not include the fraction a/b . Other terms (which perhaps better fit the definition) are:

a is a multiple of b ; and

b is a factor of a .

For example, 4 divides 12 since $12 = (4)(3)$.

It is best to follow the definition and avoid introducing fractions when using the term “divides.”

Definition 4: An integer p is prime provided $p > 1$ and the only positive divisors of p are 1 and p .

Comment: Note that there is a hidden (i.e., unnamed) variable in this definition; specifically, no symbol is given for a positive divisor of p .

An alternate form of the definition of prime is:

An integer p is prime provided $p > 1$ and for every positive integer d , if d divides p then either $d = 1$ or $d = p$.

Definition 5: An integer n is **composite** provided $n \neq 0$, $n \neq \pm 1$ and there exists integers a and b such that $n = ab$, and $a \neq \pm 1$, and $b \neq \pm 1$.

SECTION 1.4: EXERCISES

1.4.1. Consider the following definition.

Definition: An integer n is a perfect square provided there is an integer k such that $n = k^2$.

- (a) Write the given definition in symbolic form.
- (b) Give the symbolic form of the definition of: n is not a perfect square.
- (c) Write out a useful definition of: n is not a perfect square.

1.4.2. Complete the following definition:

An integer n is not even provided

1.4.3. Complete the following definition:

An integer n is not odd provided

1.4.4. Complete the following definition:

The integer b does not divide the integer a provided

1.4.5. Argue that if an integer $n > 1$ is not prime then it is a composite.

1.4.6. Given definition:

A sequence $\{x_n\}$ of real numbers is a **Cauchy sequence** provided for every positive real number ϵ there exists a natural number (that is, a positive integer) M such that for all natural numbers m and n , if $m > M$ and $n > M$ then $|x_n - x_m| < \epsilon$.

- (a) Write the given definition in symbolic form using the following assignments:

$U_{\{x_n\}}$: all sequences of real numbers.

U_ϵ : The set of all positive real numbers.

$U_M = U_m = U_n = \mathbf{Z}^+$, the set of all positive integers.

$N(\{x_n\})$: $\{x_n\}$ is a **Cauchy sequence**

$P(a, M)$: $a > M$. (So $P(m, M)$ is: $m > M$)

$Q(m, n, \epsilon)$: $|x_n - x_m| < \epsilon$

- (b) Give the symbolic form of the definition of: $\{x_n\}$ is not a Cauchy sequence.
- (c) Write out a useful definition of: $\{x_n\}$ is not a Cauchy sequence.

1.4.7. Given definition:

For a function f and real numbers a and L we say that the limit of f as x approaches a is L (hereafter written as $\lim_{x \rightarrow a} f(x) = L$) provided for every positive real number ϵ there exists a positive real number δ such that for every real number x , if $|x - a| < \delta$ then $|f(x) - L| < \epsilon$.

(a) Write the given definition in symbolic form using the following assignments:

U_f : all functions that map the reals into the reals.

$U_a = U_L = U_x$: all real numbers.

$U_\epsilon = U_\delta$: all positive real numbers

$N(f, a, L)$: $\lim_{x \rightarrow a} f(x) = L$

$P(x, a, \delta)$: $|x - a| < \delta$

$Q(f, x, L, \epsilon)$: $|f(x) - L| < \epsilon$

(b) Give the symbolic form of the definition of: $\lim_{x \rightarrow a} f(x) \neq L$.

(c) Write out a useful definition of: $\lim_{x \rightarrow a} f(x) \neq L$.