Vector Spaces; n-Dimensional Euclidean Space

Introduction  The Space $E^n$ consists of all ordered “n-tuples” of real or complex numbers

$$X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad Y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}, \quad V = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \text{(in } E^3)$$, etc.,

where $x_i, y_i$, are scalars (real or complex numbers), called the components of $X$ and $Y$, respectively. The qualification ordered means that, e.g.,

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \neq \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}.$$  

In many physical applications one wishes to restrict attention to vectors having only real components; in such cases it is common to refer to the space as $R^n$. For the most part we will consider the more general case of $E^n$; when we give a definition referring to vectors in $E^n$ it will be understood that the obvious restriction of that definition to $R^n$ applies as well, unless we explicitly state to the contrary. Vectors in $E^n$ can be thought of either as points in space or as directed magnitudes, the latter often represented graphically by arrows in the special cases of $R^2$ and $R^3$. Quantities representable by a single number $\alpha$ (e.g., speed, time, weight, density) are scalar quantities. In particular, real and complex numbers are scalars in these notes. Real numbers are quantities which have only magnitude, $|\alpha|$, and sign (+ or -). Complex numbers are really two-dimensional vectors with a particular algebraic structure; the sign is replaced by the argument of a complex number, essentially the angular component of the polar representation of the vector.
Magnitude of a Vector  If \( X = (x_1, x_2, ..., x_n) \) is an \( n \)-dimensional vector, its magnitude or norm is

\[
\|X\| = \sqrt{|x_1|^2 + |x_2|^2 + \cdots + |x_n|^2}.
\]

This is the length of a vector in the ordinary, Euclidean, sense. Thus

\[
\left\| \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix} \right\| = \sqrt{(-2)^2 + (1)^2 + (3)^2} = \sqrt{14} = \sqrt{2}\sqrt{7}.
\]

\[
\left\| \begin{pmatrix} 1 + i \\ i \\ 2 - i \end{pmatrix} \right\| = \sqrt{(1^2 + 1^2) + (0^2 + 1^2) + (2^2 + (-1)^2)} = \sqrt{8} = 2\sqrt{2}.
\]

Algebraic Operations on Vectors

The operations of addition and subtraction of vectors are performed componentwise; the result is another vector of the same size or dimension:

\[
Z = X \pm Y : \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \pm \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}, \text{ i.e., } z_k = x_k \pm y_k, k = 1, 2, ..., n.
\]

Thus, for example,

\[
\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} -3 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 - 3 \\ 0 - 1 \\ 1 + 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix};
\]

\[
\begin{pmatrix} 1 + i \\ i \\ 2 - i \end{pmatrix} + \begin{pmatrix} -3 \\ -1 \\ 1 + 2i \end{pmatrix} = \begin{pmatrix} -2 + i \\ -1 + i \\ 3 + i \end{pmatrix}.
\]

Addition or subtraction of two vectors of unequal dimension is not defined.
Component by component multiplication of vectors generally has little significance in applications but it does have some importance in respect to the Discrete Fourier Transform, defined elsewhere in these notes. The concept is more or less obvious but we need to introduce some notation. If

\[ X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad Y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \]

then \( Z = X \otimes Y \) is the \( n \)-dimensional vector such that \( z_k = x_k \cdot y_k, \ k = 1, 2, \ldots, n \).

**Scalar Multiplication** Multiplication of vectors by scalars, the latter commonly denoted by Greek letters (\( \alpha, \beta, \gamma, \ldots \), etc.) is also defined componentwise:

\[ \alpha X = \alpha \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \alpha x_1 \\ \alpha x_2 \\ \vdots \\ \alpha x_n \end{pmatrix}, \quad \text{e.g.} \quad (1 + i) \begin{pmatrix} 2 - i \\ -1 \\ 1 + 2i \end{pmatrix} = \begin{pmatrix} 3 + i \\ -1 - i \\ -1 + 3i \end{pmatrix}. \]

In particular, \( -1 X = -1 (x_1, x_2, \ldots, x_n) \equiv -X \) is the **additive inverse** of \( X \) and \( 0 X = 0 (x_1, x_2, \ldots, x_n) \equiv (0, 0, \ldots, 0) \equiv 0 \) is the **additive identity**.

**Linear Combinations of Vectors** Formation of **linear combinations** of vectors is the most important vector operation, the defining operation in the construction of **vector spaces**. In general, if \( \alpha, \beta, \gamma, \ldots \) are scalars and \( X, Y, Z, \ldots \) are vectors of the same dimension, we have

\[ \alpha X + \beta Y + \ldots + \gamma Z = \begin{pmatrix} \alpha x_1 + \beta y_1 + \ldots + \gamma z_1 \\ \alpha x_2 + \beta y_2 + \ldots + \gamma z_2 \\ \vdots \\ \alpha x_n + \beta y_n + \ldots + \gamma z_n \end{pmatrix}. \]
Thus, e.g.,
\[
-1 \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} + 2 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + 3 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} -2 + 2 + 3 \\ -1 + 2 + 0 \\ -3 + 2 - 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ -4 \end{pmatrix}.
\]

The "Dot", or "Inner", Product  If $X$ and $Y$ are vectors in $\mathbb{R}^n$, then the **dot product** or **inner product** of $X$ and $Y$ is

\[
X \cdot Y = \langle X, Y \rangle = Y^t X = \sum_{k=1}^{n} x_k y_k.
\]

Thus, e.g., $(2, -1, 3) \cdot (-1, 2, 1) = 2(-1) + (-1)2 + 3(1) = -1$. Notice that

\[
X \cdot X = (x_1)^2 + (x_2)^2 + ... + (x_n)^2 = \|X\|^2 \geq 0, \ X \neq 0.
\]

That is, the norm is naturally expressed in terms of the dot product. It is clear in this case that $X \cdot Y = Y \cdot X$.

On the other hand, for vectors in $\mathbb{E}^n$, the **dot product** or **inner product** of $X$ and $Y$ is

\[
X \cdot Y = \langle X, Y \rangle = Y^t X = \sum_{k=1}^{n} x_k y_k,
\]

where, for any complex number $z$, $\overline{z}$ denotes the **complex conjugate** of $z$; if $z = x + iy$, then $\overline{z} = x - iy$. Thus, e.g., $(1 + i, i, 2 - i) \cdot (2 - i, -1, 1 + 2i) = (1 + i)(2 + i) + i(-1) + (2 - i)(1 - 2i) = 1 - 3i$. This modified definition is adopted so that the norm of a vector $X$ as defined by

\[
\|X\| = \sqrt{X \cdot X} = \sqrt{\sum_{k=1}^{n} x_k \overline{x_k}} = \sqrt{\sum_{k=1}^{n} |x_k|^2}
\]

will be a non-negative number, corresponding to our notion of magnitude as a non-negative numerical quantity. In the complex case $X \cdot Y \neq Y \cdot X$ in general; rather, we have $Y \cdot X = \overline{X} \cdot Y$. 
Properties of the Dot Product

The *distributive* laws are:

\[(\alpha X + \beta Y) \cdot Z = \alpha(X \cdot Z) + \beta(Y \cdot Z), \quad X \cdot (\alpha Y + \beta Z) = \alpha X \cdot Y + \beta X \cdot Z.\]

These imply, in particular, that

\[(\alpha X) \cdot Y = \alpha(X \cdot Y), \quad X \cdot (\alpha Y) = \alpha X \cdot Y.\]

The *commutative* law

\[X \cdot Y = Y \cdot X\]

holds for vectors in \(R^n\); for vectors in \(E^n\) we have

\[Y \cdot X = \sum_{k=1}^{n} y_kx_k = \sum_{k=1}^{n} x_ky_k = X \cdot Y.\]

In either case we have

\[X \cdot X \geq 0, \quad X \cdot X > 0, \quad X \neq 0.\]

The *Schwarz inequality*

\[|X \cdot Y| \leq \|X\| \|Y\|.\]

is valid in either \(R^n\) or \(E^n\) with equality holding if and only if \(X, Y\) and 0 lie on the same line through the origin; i.e., if and only if there are scalars \(\alpha\) and \(\beta\) such that \(\alpha X + \beta Y = 0\). We will have more to say about this shortly.

**Corresponding Properties of the Norm**

Since \(\|X\| = \sqrt{X \cdot X} = \sqrt{\langle X, X \rangle}\) in either \(R^n\) or \(E^n\) we can readily obtain the properties of the norm from those of the dot product. We have, electing to use \(\langle , \rangle\) here,

\[\|\alpha X + \beta Y\|^2 = \langle \alpha X + \beta Y, \alpha X + \beta Y \rangle\]
\[ = |\alpha|^2 \|X\|^2 + |\beta|^2 \|Y\|^2 + \alpha \overline{\beta} \langle X, Y \rangle + \overline{\alpha} \beta \langle Y, X \rangle \]
\[ = |\alpha|^2 \|X\|^2 + |\beta|^2 \|Y\|^2 + 2 \text{Re} \left( (\alpha \overline{\beta}) \langle X, Y \rangle \right); \]
\[
\|\alpha X\| = |\alpha| \|X\|;
\]
\[
\|X\| \geq 0; \quad \|X\| = 0 \rightarrow X = 0;
\]
\[
\|X + Y\| \leq \|X\| + \|Y\|.
\]

The last is called the \textit{triangle inequality} because it is related to the familiar result from geometry to the effect that the length of any side of a triangle is less than or equal to the sum of the lengths of the other two sides; we will see shortly that it can be proved using the Schwarz inequality satisfied by the inner product.

\textbf{The Distance Between Two Points} \quad \text{This is defined as}
\[
d(X, Y) = \|X - Y\| = \|Y - X\|.
\]

The triangle inequality is so named because
\[
d(X, Y) = \|X - Y\| = \|X - Z + Z - Y\| \leq \|X - Z\| + \|Z - Y\|
\]
\[
= d(X, Z) + d(Z, Y)
\]
for any three points (forming the vertices of a triangle) in \( E^n \).
Proposition 1  Let $X, Y$ be vectors in $\mathbb{R}^n$ and let $\theta$ denote the angle between them. Then

$$X, Y = \|X\| \|Y\| \cos \theta.$$ 

Moreover the inequality reduces to an equality just in case $X$ and $Y$ are collinear, i.e., there are scalars $\alpha$ and $\beta$, not both zero, such that $\alpha X + \beta Y = 0$.

We will present two proofs of this proposition. The first, valid in $\mathbb{R}^n$, is an immediate consequence of a familiar result from trigonometry.

Proof of the Schwarz inequality in $\mathbb{R}^n$  From the Law of Cosines we have

$$\|X - Y\|^2 = \|X\|^2 + \|Y\|^2 - 2 \|X\| \|Y\| \cos \theta.$$ 

On the other hand

$$\|X - Y\|^2 = \langle X - Y, X - Y \rangle = \langle X, X \rangle + \langle Y, Y \rangle - 2 \langle X, Y \rangle.$$ 

Comparing these two identities we immediately have the Schwarz inequality in $\mathbb{R}^n$:

$$|X \cdot Y| = \|X\| \|Y\| |\cos \theta| \leq \|X\| \|Y\| \ 1 = \|X\| \|Y\|.$$ 

The inequality becomes an equality just in case $\cos \theta = 0$, in which case $X$ and $Y$ clearly lie on the same line through the origin. This completes the proof.

The proof in $E^n$ is only slightly more complicated than the previous one but lacks its intuitive simplicity. It is also valid when specialized to the real case.
Proof of the Schwarz Inequality in $E^n$  Since the inequality is obvious if either $X$ or $Y$ is the zero vector, let us suppose neither $X$ nor $Y$ is the zero vector. Then suppose first of all that $\|X\| = 1$; $\|Y\| = 1$. For any complex scalars $\alpha$ and $\beta$ we have

$$\|\alpha X - \beta Y\|^2 = |\alpha|^2 \|X\|^2 + |\beta|^2 \|Y\|^2 - \alpha \overline{\beta} X \cdot Y - \overline{\alpha \beta} Y \cdot X.$$  

Taking $\alpha = Y \cdot X$ and $\beta = 1$ and remembering $\|X\| = 1$; $\|Y\| = 1$ we obtain

$$\|\alpha X - \beta Y\|^2 = |Y \cdot X|^2 + 1 - 2|X \cdot Y|^2 = 1 - |X \cdot Y|^2.$$  

Since the left hand side must be non-negative, the right hand side must also be non-negative and we have

$$|X \cdot Y|^2 \leq 1 \Rightarrow |X \cdot Y| \leq 1 (= \|X\| \|Y\|).$$  

Then, for arbitrary non-zero $X$ and $Y$ we must have

$$\left| \frac{X}{\|X\|} \cdot \frac{Y}{\|Y\|} \right| \leq 1 \Rightarrow |X \cdot Y| \leq \|X\| \|Y\|.$$  

Clearly the only way in which the inequality can be an equality is if $\alpha X + \beta Y = 0$. This completes the proof.

Proposition 2  The triangle inequality follows from the Schwarz inequality.

$$\|X + Y\|^2 = \langle X + Y, X + Y \rangle = \|X\|^2 + \|Y\|^2 + 2 \langle X, Y \rangle \leq \|x\|^2 + \|y\|^2 + 2 \|x\| \|y\| - (\|x\| + \|y\|)^2$$

$$\Rightarrow \|X + Y\| \leq \|X\| + \|Y\|.$$  

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Orthogonal and Orthonormal Vectors  Two vectors $X$ and $Y$ are said to be (mutually) orthogonal if $X \cdot Y = 0$. In the real case the law of cosines shows this to imply that the angle between the vectors $X$ and $Y$ is a right angle, provided neither $X$ nor $Y$ is the zero vector. (A trivial instance of orthogonality occurs if either $X = 0$ or $Y = 0$; in this case the angle between the two vectors is not defined. A vector $U$ is a unit vector if $\|U\| = 1$. For any non-zero vector $Q$ the normalization of $Q$ is the unit vector $U_Q = \frac{Q}{\|Q\|}$.

A set of vectors $U_1$, $U_2$, ..., $U_n$ forms an orthogonal set if

$$U_k \neq 0, k = 1, 2, ..., n; \langle U_k, U_j \rangle = 0, k \neq j.$$  

They form an orthonormal set if

$$\|U_k\| = 1, k = 1, 2, ..., n; \langle U_k, U_j \rangle = 0, k \neq j.$$  

Examples  First we give some examples in $\mathbb{R}^3$:

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix};$$

$$\begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}, \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{pmatrix}, \begin{pmatrix} -1/\sqrt{6} \\ 2/\sqrt{6} \\ -1/\sqrt{6} \end{pmatrix}.$$  

Now an example from $\mathbb{E}^4$; note that it is essential to use the conjugate in $X \cdot Y = \sum_{k=1}^n x_k\overline{y_k}$ in order to obtain the orthonormality.

$$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ i \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -i \\ 1 \\ i \end{pmatrix}.$$
Component of One Vector with Respect to Another  Suppose $Q$ is a non-zero vector and $X$ is an arbitrary vector, both in $E^n$. The (vector) component of $X$ in the direction of $Q$ is a scalar multiple of $Q$, $rQ$, such that the angle $\psi$ between the vectors $Q$ and $X - rQ$ is a right angle. Thus

$$Q \cdot (X - rQ) = 0 \Rightarrow r \|Q\|^2 = X \cdot Q$$

$$\Rightarrow r = \frac{X \cdot Q}{\|Q\|^2} \Rightarrow rQ = \frac{X \cdot Q}{\|Q\|^2} Q.$$

Another way to write this is

$$rQ = \left( X \cdot \frac{Q}{\|Q\|} \right) \frac{Q}{\|Q\|} = (X \cdot U_Q) U_Q.$$

It is clear that the component of $X$ with respect to $Q$ is the zero vector just in case $X$ and $Q$ are mutually orthogonal.

Example  Let $Q = (1, 2, 3)$ and let $X = (3, 2, 1)$, $Y = (1, -2, 1)$. Then $X \cdot Q = 10$, $Y \cdot Q = 0$. The components of $X$ and $Y$ with respect to $Q$ are

$$\frac{X \cdot Q}{\|Q\|^2} Q = \frac{10}{14} Q = \frac{5}{7} Q$$

and

$$\frac{Y \cdot Q}{\|Q\|^2} Q = \frac{0}{14} Q = 0,$$

respectively.

Problems

1. Form the vector $X$ which is the linear combination in $E^3$:

$$X = (1 + i) \begin{pmatrix} \frac{1}{2 + \sqrt{3}} \\ \frac{1}{2 - \sqrt{3}} \end{pmatrix} + (1 - i) \begin{pmatrix} \frac{1}{2 - \sqrt{3}} \\ \frac{1}{2 + \sqrt{3}} \end{pmatrix}.$$

2. (a) Find a vector $Z$ in $E^3$ which is orthogonal to both of the vectors

$$X = \begin{pmatrix} \frac{1}{2 + \sqrt{3}} \\ \frac{1}{2 - \sqrt{3}} \end{pmatrix}; \ Y = \begin{pmatrix} \frac{1}{2 - \sqrt{3}} \\ \frac{1}{2 + \sqrt{3}} \end{pmatrix}.$$
(b) Is there only one such vector \(X\)?

(c) Same question as in (b), but require in addition that \(\|X\| = 1\).

(d) Compute \(X \cdot Y\), where \(X\) and \(Y\) are the vectors in part (a).

3. Does the vector \((-\frac{1}{2}, 1, -\frac{1}{2})\) lie in the span of the vectors \((1, -1, 0)\) and \((0, 1, -1)\)?

4. Normalize the vector \((1, 1+i, 1-i)\).

5. Suppose the vectors \(X, Y, Z\) form an orthonormal set in \(\mathbb{R}^3\) and

\[
X = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} ; \quad Y = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}.
\]

There are only two choices for \(Z\). What are they?