Name:____________________

Math 4426; Project 1; Due: 2-28-08

In this project we construct an approximate solution $u(x,\, t)$ for the PDE

$$\rho(x)\,\frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x}\left(p(x)\frac{\partial u}{\partial x}\right) = 0,\;\; t \geq 0,\;\; 0 \leq x \leq 5,$$

with boundary and initial conditions

$$u(0,\, t) = 0,\;\; \frac{\partial u}{\partial x}(5,\, t) = 0, \quad u(x,\, 0) = 0,\;\; \frac{\partial u}{\partial t}(x,\, 0) = 1.$$

The coefficients are given by

$$\rho(x) = 1 - x/6,\;\; p(x) = 10\,(1 - x/6)\,.$$

We divide the interval $[0,\, 5]$ into 20 subintervals of length $h = 1/4$ by means of $N + 1 = 21$ equally spaced points $x_k,\;\; k = 0, 1, 2, ..., 20$, where the solution is represented by corresponding time functions $u_j(t)$; $u_0(t) \equiv 0$. We set

$$\rho_k = \rho(x_k),\;\; p_k = p\left(\frac{x_k + x_{k-1}}{2}\right),\;\; k = 1, 2, ..., 20.$$

We construct the matrix

$$\mathbf{A} = \frac{1}{h^2}\begin{pmatrix} -\frac{p_1+p_2}{\rho_1} & \frac{p_2}{\rho_1} & 0 & \cdots & 0 & 0 \\ \frac{p_2}{\rho_2} & -\frac{p_2+p_3}{\rho_2} & \frac{p_3}{\rho_2} & \cdots & 0 & 0 \\ 0 & \frac{p_3}{\rho_3} & -\frac{p_3+p_4}{\rho_3} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -\frac{p_{N-1}+p_N}{\rho_{N-1}} & \frac{p_N}{\rho_{N-1}} \\ 0 & 0 & 0 & \vdots & \frac{p_N}{\rho_N} & -\frac{p_N}{\rho_N} \end{pmatrix}$$

and, using the Matlab operation >> $[X,\, D] = \mathrm{eig}(A)$ we obtain the eigenvalues of $\mathbf{A}$, which are negative, as the diagonal elements, $d_k \equiv -\omega_k^2$, of D and the eigenvectors, X_k, as the corresponding columns of the matrix X.

For the project, compute X and D as indicated and verify that the eigenvectors X_k are ρ- orthogonal, i.e., $\sum_{\ell=1}^{N} \rho_k \, X_{k,\ell} X_{j,\ell} = 0, \;\; k \neq j$. Expand the initial velocity vector $V_0, \;\; V_{0,k} = 1, \;\; k = 1, 2, ..., N$, in terms of the X_k, via

$$V_0 = \sum_{k=1}^{N} v_k \, X_k, \qquad v_k = \frac{\langle V_0, X_k \rangle_\rho}{\|X_k\|_\rho^2}.$$

The approximate solution of the PDE then corresponds to the $N+1$-dimensional vector function $U(t)$ with components

$$u_0(t) \equiv 0, \;\; u_j(t) = \sum_{k=1}^{N} \frac{v_k}{\omega_k} \sin(\omega_k t) \; X_{k,j}, \;\; j = 1, 2, ..., N.$$

For your report, list the eigenvalues $d_k, \;\; k = 1, 2, ..., N(= 20)$ and plot the eigenvector components $X_{k,j}$ versus $x_j, \;\; j = 1, 2, ..., N$, all on the same diagram but appropriately labelled, using gtext.m, or equivalent, for $k = 1, 2, ..., N/2(= 10)$. Then use surf.m, mesh.m, or equivalent, to plot the approximate solution

$$u(0, t_\ell) = 0, \;\; u(x_j, t_\ell) \approx u_j(t_\ell),$$

versus the points

$$(0, t_\ell), \;\; (x_j, t_\ell), \;\; j = 1, 2, ..., N(= 20), \;\; \ell = 0, 1, 2, ..., 200, \;\; t_\ell = .1\,\ell.$$

All plots should be done using standard computer software and your report should be printed. Include any comments necessary in order to explain your work. Attach these pages to the front of your report, with your name written at the top.