Sine and Cosine Series; Odd and Even Functions

A sine series on the interval $[0, L]$ is a trigonometric series of the form
\[ \sum_{k=1}^{\infty} b_k \sin \frac{\pi k x}{L}. \]
All of the terms in a series of this type have values vanishing at $x = 0$ and at $x = L$. Such series are useful in representing functions $f(x)$ which satisfy boundary conditions of the form $f(0) = f(L) = 0$. By adding a constant function it is possible to represent functions $f(x)$ satisfying $f(0) = f(L) = c$ for any given $c$ and by adding a linear function of the form $\frac{1}{L} (c(L - x) + dx)$ it is possible to represent functions satisfying boundary conditions of the form $f(0) = c, f(L) = d$. These properties make sine series very useful in a variety of applications, including solution of the heat equation and the wave equation.

A cosine series is a trigonometric series of the form
\[ \sum_{k=0}^{\infty} a_k \cos \frac{\pi k x}{L}. \]
All of the terms in a series of this type are such that their derivatives vanish at $x = 0$ and at $x = L$, making them useful in the representation of functions $f(x)$ satisfying boundary conditions of the form $f'(0) = f'(L) = 0$. By adding functions of the form $d \cdot x$, $d$ constant, functions satisfying conditions $f'(0) = f'(L) = d$ can be constructed.

But what reason do we have to suppose that general functions should be expandable in series of these types? In order to see that this is, in fact, to be expected it is desirable to consider certain ways in which a function $f(x)$ initially defined on an interval $[0, L]$ can be extended beyond that interval.

Definition 1 A function $f(x)$ defined on a domain which includes an interval $[-L, L], L > 0$, is an even function on that interval if for all $x$ there we have $f(-x) = f(x)$. It is an odd function on that interval if $f(-x) = -f(x)$ there. (The positive number $L$ can be replaced by $\infty$, in which case the interval in question is $(-\infty, \infty)$ and we...
can call the function even or odd without any qualifying statement about the interval involved.)

There are numerous examples of even functions: \( \cos x, \ x^2, \ \frac{1}{1+x^2}, \ \cosh x, \ x^4, \ \ldots \), etc., etc.. Odd functions include \( \sin x, \ x, \ \frac{x}{1+x^2}, \ \sinh x, \ x^3, \ \ldots \), etc., etc.. It is easy to see that the product of two even functions, or the product of two odd functions, is an even function whereas the product of an even function and an odd function is an odd function. The following proposition is often useful.

**Proposition 1** Let \( f(x) \) be piecewise continuous on an interval \([-L, L], \ L > 0\). Then there is a function \( g(x) \) even on \([-L, L]\) and a function \( h(x) \) odd on \([-L, L]\) such that \( f(x) = g(x) + h(x) \), \( x \in [-L, L] \).

**Proof** \( g(x) = \frac{1}{2}(f(x) + f(-x)) \), \( h(x) = \frac{1}{2}(f(x) - f(-x)) \), \( x \in [-L, L] \).

It is clear that \( g(x) + h(x) = f(x) \). To show that \( g(x) \) is even on \([-L, L]\) we let \( x \) lie in that interval and we compute

\[
    g(-x) = \frac{1}{2}(f(-x) + f(-(-x))) = \frac{1}{2}(f(-x) + f(x)) = g(x);
\]
a similar argument shows that \( h(x) \) is odd on \([-L, L]\), completing the proof.

If \( f(x) \) is an arbitrary piecewise continuous function defined on the interval \([0, L], \ L > 0\), we can extend the definition of this function to the interval \([-L, L]\) in any number of ways but two of these are particularly significant for our purposes. The even extension, \( f_e(x) \), is obtained by defining

\[
    f_e(x) = \begin{cases} 
    f(x), & x \in [0, L] \\
    f(-x), & x \in [-L, 0) 
  \end{cases}.
\]

For example, the even extension of \( f(x) \equiv x \), initially defined on an interval \([0, L] \) \( L > 0 \), is \( f_e(x) = |x| \), \( x \in [-L, L] \).

The odd extension, \( f_o(x) \), of \( f(x) \) is obtained by setting

\[
    f_o(x) = \begin{cases} 
    f(x), & x \in [0, L] \\
    -f(-x), & x \in [-L, 0) 
  \end{cases}.
\]
If \( f(0) \), as originally defined on \([0, L]\), is not 0, the extended function will have a jump discontinuity at that point. To preserve odd symmetry it may be desirable to redefine \( f(0) = 0 \) in the extended function but that is immaterial for most purposes. The odd extension of \( f(x) \equiv 1, \ x \in [0, \infty) \) may thus be taken to be the function
\[
f_o(x) = \begin{cases} 
1, & x > 0, \\
0, & x = 0, \\
-1, & x < 0, 
\end{cases}
\]
which is often called the sign function or \( \text{sgn}(x) \).

The cosine series for a function \( f(x) \) defined on \([0, L]\) is obtained by first constructing the even extension \( f_e(x) \) of \( f(x) \) to the interval \([-L, L]\). We then form the real Fourier series for the extended function \( f_e(x) \). Since the length of the extended interval is \( 2L \) rather than \( L \), that series takes the form
\[
a_0 + \sum_{k=1}^{\infty} \left( a_k \cos \frac{2\pi k x}{2L} + b_k \sin \frac{2\pi k x}{2L} \right) = a_0 + \sum_{k=1}^{\infty} \left( a_k \cos \frac{\pi k x}{L} + b_k \sin \frac{\pi k x}{L} \right).
\]
The coefficients in this Fourier series are given by
\[
a_0 = \frac{1}{2L} \int_{-L}^{L} f_e(x) \, dx = \frac{1}{L} \int_{0}^{L} f(x) \, dx,
\]
\[
a_k = \frac{2}{2L} \int_{-L}^{L} \cos \frac{2\pi k x}{2L} f_e(x) \, dx = \frac{2}{L} \int_{0}^{L} \cos \frac{\pi k x}{L} f(x) \, dx,
\]
\[
b_k = \frac{2}{2L} \int_{-L}^{L} \sin \frac{2\pi k x}{2L} f_e(x) \, dx = 0.
\]
The second equality in the formulas for \( a_0 \) and the \( a_k \) follows from the fact that \( f_e(x) \) is even and, also, \( \cos \frac{\pi k x}{L} f_e(x) \) is even, being the product of two even functions. In the case of the formula for the \( b_k \), the function \( \sin \frac{\pi k x}{L} f_e(x) \), being the product of an odd function and an even function, is an odd function and therefore its integral over \([-L, L]\) must be zero. Thus we have the cosine series representations
\[
f(x) = a_0 + \sum_{k=1}^{\infty} a_k \cos \frac{\pi k x}{L}, \ x \in [0, L],
\]
\[
f_e(x) = a_0 + \sum_{k=1}^{\infty} a_k \cos \frac{\pi k x}{L}, \ x \in [-L, L]
\]
with \( a_0 \) and \( a_k \) given in terms of \( f(x), \ x \in [0, L] \), as indicated.
An entirely similar process, using the odd extension \( f_o(x) \) of \( f(x) \) to \([-L, L]\), which we need not repeat in detail, leads to sine series representation

\[
f(x) = \sum_{k=1}^{\infty} b_k \sin \frac{\pi k x}{L}, \quad x \in [0, L],
\]

\[
f_o(x) = \sum_{k=1}^{\infty} b_k \sin \frac{\pi k x}{L}, \quad x \in [-L, L]
\]

with the coefficients \( b_k \) given by

\[
b_k = \frac{2}{L} \int_0^L \sin \frac{\pi k x}{L} f(x) \, dx.
\]

It should be noted that on \([0, L]\) we have

\[
f(x) = a_0 + \sum_{k=1}^{\infty} a_k \cos \frac{\pi k x}{L}
\]

and we also have, on that same interval

\[
f(x) = \sum_{k=1}^{\infty} b_k \sin \frac{\pi k x}{L};
\]

the single function \( f(x) \) on the original interval \([0, L]\) has both a cosine series and a sine series representation. However, when these series are extended to the interval \([-L, L]\), they represent the distinct functions \( f_e(x) \) and \( f_o(x) \), respectively.

**Example 1** Let us consider the function \( f(x) \equiv 1, \ 0 \leq x \leq L, \ L > 0 \). Then the coefficients of the sine series expansion are

\[
b_k = \frac{2}{L} \int_0^L \sin \frac{\pi k x}{L} \cdot 1 \, dx = - \frac{2}{\pi k} \cos \frac{\pi k x}{L} \bigg|_0^L = \left\{ \begin{array}{ll} \frac{4}{\pi k}, & k \text{ odd,} \\ 0, & k \text{ even.} \end{array} \right.
\]

Now we take the same function, \( f(x) \equiv 1, \ 0 \leq x \leq L \), but we expand it in cosine series. Now the coefficients are, almost trivially,

\[
a_0 = \frac{1}{L} \int_0^L 1 \, dx = 1, \quad a_k = \frac{2}{L} \int_0^L \cos \frac{\pi k x}{L} \cdot 1 \, dx = \frac{2}{\pi k} \sin \frac{\pi k x}{L} \bigg|_0^L = 0, \ k \geq 1.
\]
Example 2  We consider the function $f(x) = x, \ x \in [0, \pi]$. To compute the coefficients of the sine series, we form, for $k = 1, 2, 3, \ldots$,

$$b_k = \frac{2}{\pi} \int_0^\pi \sin kx \cdot x \, dx = -\frac{2}{k\pi} \cos k\pi \bigg|_0^\pi + \frac{2}{k\pi} \int_0^\pi \cos kx \, dx$$

$$= -\frac{2}{k} \cos k\pi + \frac{2}{k^2 \pi} \sin k\pi \bigg|_0^\pi = (-1)^{k+1} \frac{2}{k},$$

Thus we have, in the sense of convergence with respect to the norm in $L^2[0, \pi]$,

$$x = \sum_{k=1}^\infty (-1)^{k+1} \frac{2}{k} \sin kx.$$ 

In this case the series also represents $f(x) = x$, in the same sense, on the interval $[-\pi, 0]$ because $f(x) = x$ is an odd function and thus equal to its own odd extension $f_o(x)$.

To develop the same function in cosine series we compute

$$a_0 = \frac{1}{\pi} \int_0^\pi x \, dx = \frac{1}{\pi} \frac{x^2}{2} \bigg|_0^\pi = \frac{\pi}{2},$$

and, for $k = 1, 2, 3, \ldots$,

$$a_k = \frac{2}{\pi} \int_0^\pi \cos kx \cdot x \, dx = \frac{2}{k\pi} \sin k\pi \bigg|_0^\pi - \frac{2}{k\pi} \int_0^\pi \sin kx \cdot 1 \, dx$$

$$= \frac{2}{\pi k^2} \cos k\pi \bigg|_0^\pi = \frac{2}{\pi k^2} \left((-1)^k - 1\right) = \begin{cases} \frac{4}{\pi k^2}, & k \text{ odd}, \\ 0, & \text{otherwise}. \end{cases}$$ 

In this case the series, which with $k = 2j + 1$ can be rewritten as

$$x = \frac{\pi}{2} - \sum_{j=0}^\infty \frac{4}{\pi(2j+1)^2} \cos(2j+1)x,$$

represents $-x$ on the interval $[-\pi, 0]$ because the even extension, $f_e(x)$, of $f(x) = x$ to $[-\pi, \pi]$ is $|x|$, which agrees with $-x$ on $[-\pi, 0]$.

As an application we consider the heat equation

$$\frac{\partial T}{\partial t} - \alpha \frac{\partial^2 T}{\partial x^2} = 0, \quad t \geq 0, \quad 0 \leq x \leq 1,$$

with initial and boundary conditions

$$T(x, 0) = 1, \quad 0 \leq x \leq 1, \quad T(0, t) = 0, \quad T(1, t) = 0, \quad t > 0.$$
We expand \( T(x, 0) = 1 \) using 20 terms of the sine series as shown in Example 1. Then, as in our section on solution of the heat equation, we have

\[
T(x, t) = \sum_{j=0}^{20} \frac{4}{\pi (2j + 1)} \exp \left( -\alpha \frac{2}{\pi^2} k^2 t \right) \sin (\pi k x).
\]

The figure below shows the sine approximation to the initial state \( T(x, 0) = 1 \) and plots of \( T(x, t) \) for a sequence of positive values of \( t \). Even though the boundary conditions are not consistent with the initial state \( T(x, 0) = 1 \) the use of sine series assures satisfaction of those boundary conditions for \( t > 0 \).