Fourier Transform Applications

The Fourier transform is very useful in solving a variety of linear constant coefficient ordinary and partial differential equations describing processes which take place over an infinite interval, $-\infty < x < \infty$. We will provide a number of examples of this sort of application in the present section. Our first example involves a simple, time independent, equilibrium process.

**Example 1** We consider a stretched string, or cord, with small transverse displacement $y(x)$, subject to an external transverse force $f(x)$ and a transverse restoring force $-\kappa y(x)$, maintained at tension $\tau > 0$ over the interval $-\infty < x < \infty$ and constrained so that $\lim_{|x| \to \infty} y(x) = 0$. It can then be shown that $y(x)$ satisfies

$$\tau \frac{d^2y}{dx^2} - \kappa y(x) + f(x) = 0.$$  

Taking $a^2 = \sqrt{\kappa/\tau}$ and applying the Fourier transform to both sides of this equation, using the differentiation property (twice) we have

$$- (\xi^2 + a^2) \hat{y}(\xi) + \frac{1}{\tau} \hat{f}(\xi) = 0 \Rightarrow \hat{y}(\xi) = \frac{1}{\tau \xi^2 + a^2} \hat{f}(\xi).$$

Using the convolution property of the Fourier transform we obtain

$$y(x) = \frac{1}{\tau} \int_{-\infty}^{\infty} \left( \mathcal{F}^{-1} \frac{1}{\xi^2 + a^2} \right)(r) f(x - r) \, dr$$

as the solution.

To make any further progress on this we need to find the function $K(x)$ such that $\hat{K}(\xi) \equiv (\mathcal{F}K)(\xi) = \frac{1}{\xi^2 + a^2}$. It turns out that this function is

$$K(x) = \begin{cases} 
\frac{e^{-ax}}{2a}, & x > 0, \\
\frac{e^{ax}}{2a}, & x < 0.
\end{cases}$$
To see that this is the right function we compute
\[
\int_{-\infty}^{0} e^{-i\xi x} e^{ax} \, dx + \int_{0}^{\infty} e^{-i\xi x} e^{-ax} \, dx
\]
\[
= \int_{-\infty}^{0} e^{(a-i\xi)x} \, dx + \int_{0}^{\infty} e^{-(a+i\xi)x} \, dx = \left( \frac{1}{a - i\xi} + \frac{1}{a + i\xi} \right) = \frac{2a}{a^2 + \xi^2}.
\]
Thus \((\mathcal{F}K(x))(\xi) = \frac{1}{a^2 + \xi^2}\) as claimed. Going back to the earlier formula for \(y(x)\) we now have
\[
y(x) = \frac{1}{\sqrt{\kappa\tau}} \left( \int_{-\infty}^{0} e^{ar} f(x - r) \, dr + \int_{0}^{\infty} e^{-ar} f(x - r) \, dr \right).
\]

We can also use this example to make another point. We have
\[
\frac{1}{a^2 + \xi^2} = \int_{-\infty}^{\infty} e^{-i\xi x} K(x) \, dx.
\]
Interchanging the roles of \(\xi\) and \(x\),
\[
\frac{1}{a^2 + x^2} = \int_{-\infty}^{\infty} e^{-i\xi x} K(\xi) \, d\xi
\]
and then replacing \(x\) by \(-x\) we have
\[
\frac{1}{a^2 + x^2} = \int_{-\infty}^{\infty} e^{i\xi x} K(\xi) \, d\xi = 2\pi \left( \mathcal{F}^{-1}K(\xi) \right)(x).
\]
Applying the Fourier transform to both sides we obtain
\[
2\pi K(\xi) = \left( \mathcal{F} \frac{1}{a^2 + x^2} \right)(\xi) = \int_{-\infty}^{\infty} e^{-i\xi x} \frac{1}{a^2 + x^2} \, dx.
\]
Noting the two part expression for \(K(\xi)\), this example illustrates the point that the Fourier transform of a function having a simple algebraic expression, in this case \(\frac{1}{x^2 + a^2}\), may turn out not to be expressible in terms of a single elementary function. In other cases the Fourier transform, even where it does exist, may have no expression in terms of any number of elementary functions.
Example 2: Heat Conduction in an Infinite Rod  

We consider an infinite bar of heat conducting material, parametrized by the co-ordinate $x$, $-\infty < x < \infty$. We will suppose the specific heat per unit length is the constant $\sigma$ and the heat conductivity coefficient is $\kappa$. Denoting the temperature at the point $x$ and time $t$ by $u(x, t)$, it may be shown that the partial differential equation regulating the evolution of this temperature distribution is

$$\sigma \frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2}.$$

We will suppose that at time $t = 0$ the initial temperature distribution is $u(x, 0) = u_0(x)$; we will assume $u_0(x)$ is Fourier transformable and that the subsequent time varying heat distribution $u(x, t)$ is Fourier transformable as well. Applying the Fourier transform to both sides of the partial differential equation and using the differentiation property of the transform, we have, for $\hat{u}(\xi, t) = (\mathcal{F}u(x, t))(\xi)$,

$$\frac{\partial \hat{u}}{\partial t} = \frac{\kappa}{\sigma} \left( \mathcal{F} \frac{\partial^2 u}{\partial x^2}(x, t) \right)(\xi) = -\frac{\kappa \xi^2}{\sigma} \hat{u}(\xi, t).$$

This is a first order linear differential equation, parametrized by $\xi$, in $\hat{u}(\xi, t)$. Taking account of the given initial distribution we have

$$\hat{u}(\xi, t) = e^{-\frac{\kappa \xi^2}{\sigma} t} \hat{u}_0(\xi), \quad t \geq 0, \quad -\infty < \xi < \infty,$$

where $\hat{u}_0(\xi)$ is the Fourier transform of the initial state $u_0(x)$. Next applying the inverse Fourier transform to recover $u(x, t)$ we obtain, using the convolution property of the transform,

$$u(x, t) = \left( \mathcal{F}^{-1} \left( e^{-\frac{\kappa \xi^2}{\sigma} t} \right) \right)(x) \ast u_0(x) = \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\xi x} e^{-\frac{\kappa \xi^2}{\sigma} t} d\xi \right) \ast u_0(x).$$

For the evaluation of the integral we proceed as in Example 1 in the Fourier transform properties section. We have, for $t > 0$,

$$\int_{-\infty}^{\infty} e^{i\xi x} e^{-\frac{\kappa \xi^2}{\sigma} t} d\xi = e^{-\frac{\alpha x^2}{4\kappa t}} \int_{-\infty}^{\infty} e^{-\frac{\kappa t}{\sigma} (\xi - (\sigma ix/2\kappa t))^2} d\xi.$$
\[
= e^{-\sigma^2/4\kappa t} \int_{-\infty}^{\infty} e^{-\kappa t \xi^2} d\xi = e^{-\sigma^2/4\kappa t} \left( \int_{0}^{2\pi} \int_{0}^{\infty} e^{-\kappa t r^2} r dr d\theta \right)^{1/2} \\
= e^{-\sigma^2/4\kappa t} \left( \frac{\pi \sigma}{\kappa t} \int_{0}^{\infty} e^{-\kappa t r^2} r dr d\theta \right)^{1/2} = \sqrt{\frac{\pi \sigma}{4\pi \kappa t}} e^{-\sigma^2/4\kappa t}.
\]

Then we have the solution \( u(x, t), t > 0, \) in the form of the convolution integral

\[
u(x, t) = \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\xi x} e^{-\kappa t \xi^2} d\xi \right) * u_0(x) = \sqrt{\frac{\sigma}{4\pi \kappa t}} \int_{-\infty}^{\infty} e^{-\sigma y^2/4\kappa t} u_0(x-y) dy.
\]

It will be observed, in both Example 1 and Example 2, that the solution is ultimately expressed as the convolution of a certain kernel function with the given data. This is typical of spatially uniform processes where the effect of data given at \( x_1 \) upon the solution at \( x_2 \) depends only on the difference \( x_1 - x_2 \). This pattern continues in the next example.

**Example 3: Solution of the Wave Equation**  Plane vibrations of a stretched string, or cable, of effectively infinite length (what that means would clearly depend on the context) can be modelled by the linear partial differential equation, known as the wave equation,

\[
\rho \frac{\partial^2 u}{\partial t^2} - \tau \frac{\partial^2 u}{\partial x^2} = f(x, t),
\]

where \( \rho \) is the mass per unit length, \( \tau \) is the tension and \( f(x, t) \) is an applied external lateral force distribution, all measured in appropriate units which we do not need to specify here. We will treat the homogeneous, unforced, case wherein \( f(x, t) \equiv 0 \). The solution \( u(x, t) \) is then completely determined by the initial conditions

\[
u(x, 0) = u_0(x), \quad \frac{\partial u}{\partial t}(x, 0) = v_0(x), \quad -\infty < x < \infty,
\]

where \( u_0(x) \in L^2(-\infty, \infty) \) is the initial lateral displacement of the string and \( v_0(x) \in L^2(-\infty, \infty) \) is the initial velocity in the transverse direction.
Letting \( \hat{u}(\xi, t) \), \( \hat{u}_0(\xi) \) and \( \hat{v}_0(\xi) \) denote the Fourier transforms of \( u(x, t) \), \( u_0(x) \) and \( v_0(x) \), respectively, application of the transform to the partial differential equation, together with use of the differentiation property of the transform, yields

\[
\rho \frac{\partial^2 \hat{u}}{\partial t^2} + \tau \xi^2 \hat{u} = 0.
\]

This is a first order linear ordinary differential equation in \( \hat{u}(\xi, t) \), parametrized by \( \xi \). The general solution is

\[
\hat{u}(\xi, t) = c_1(\xi) \cos \sigma \xi t + c_2(\xi) \sin \sigma \xi t,
\]

where \( \sigma = \sqrt{\frac{\tau}{\rho}} \). Matching this solution form to the transforms \( \hat{u}_0(\xi) \), \( \hat{v}_0(\xi) \) of the initial data we obtain \( c_1(\xi) = \hat{u}_0(\xi) \), \( c_2(\xi) = \frac{\hat{v}_0(\xi)}{\sigma \xi} \), so that

\[
\hat{u}(\xi, t) = \hat{u}_0(\xi) \cos \sigma \xi t + \frac{\hat{v}_0(\xi)}{\sigma \xi} \sin \sigma \xi t.
\]

Now

\[
\cos \sigma \xi t \hat{u}_0(\xi) = \frac{1}{2} \left( e^{-i \sigma \xi t} + e^{-i \sigma \xi (-t)} \right) \hat{u}_0(\xi)
\]

\[
= \frac{1}{2} \left( (\mathcal{F}u_0(x - \sigma t)) (\xi) + (\mathcal{F}u_0(x + \sigma t)) (\xi) \right).
\]

On the other hand \( \frac{\sin \sigma \xi t}{\sigma \xi} \) is the Fourier transform of the function \( \frac{1}{2\sigma} \chi[-\sigma t, \sigma t](x) \), where

\[
\chi[-\sigma t, \sigma t](x) = \begin{cases} 1, & |x| \leq \sigma t, \\ 0, & \text{otherwise}, \end{cases}
\]

as we can see from

\[
\int_{-\sigma t}^{\sigma t} \frac{1}{2\sigma} e^{-ix} dx = -\frac{e^{-i\sigma x}}{2i\sigma \xi} \bigg|_{-\sigma t}^{\sigma t} \\
= \frac{1}{2i\sigma \xi} (e^{i\sigma \xi t} - e^{-i\sigma \xi t}) = \frac{\sin \sigma \xi t}{\sigma \xi}.
\]
We conclude, using the convolution property of the Fourier transform, that \( \hat{\phi}_0(\xi) \sin \sigma \xi t \) is the Fourier transform of the convolution

\[
\frac{1}{2\sigma} \chi_{[-\sigma t, \sigma t]}(x) * v_0(x) = \frac{1}{2\sigma} \int_{-\sigma t}^{\sigma t} v_0(x - y) \, dy.
\]

Combining these results we see that

\[
\hat{u}(\xi, t) = \frac{1}{2} \left( (\mathcal{F}u_0(x - \sigma t))(\xi) + (\mathcal{F}u_0(x + \sigma t))(\xi) \right)
\]

\[
+ \left( \mathcal{F} \left( \frac{1}{2\sigma} \int_{-\sigma t}^{\sigma t} v_0(x - y) \, dy \right) \right)(\xi)
\]

and, applying the inverse transform, we have d’Alembert’s formula:

\[
u(x, t) = \frac{1}{2} (\nu_0(x - \sigma t) + \nu_0(x + \sigma t)) + \frac{1}{2\sigma} \int_{-\sigma t}^{\sigma t} \nu_0(x - y) \, dy.
\]

**Example 4: The Curious Case of the Elastically Suspended Catwalk**

Let us suppose that a catwalk, which we model as an elastic beam, is suspended from a rigid support, say a very strong steel girder, by means of elastic stringers, or springs. Assuming the distance traversed is quite long with respect to any other dimensions involved, we can do the modelling on an infinite interval \(-\infty < x < \infty\). We will suppose the catwalk supports a point mass \( m \) at \( x = 0 \) and the gravitational constant is \( g \). Introducing the distribution \( \delta_0 \), as in the section on Laplace transforms of distributions, the vertical deflection \( u(x) \) of the catwalk from its nominal equilibrium level can be modelled by

\[
EI \frac{\partial^4 u}{\partial x^4} = -\kappa u - mg \delta_0,
\]

where \( EI \) is the *bending moment* of the elastic beam and \( \kappa \) is the stiffness, per unit beam length, of the elastic supporting structure.
Just as in our discussion of the Laplace transform one can form the Fourier transform of $\delta_0$ and one can see that $(\mathcal{F}\delta_0)(\xi) \equiv 1$. Applying the Fourier transform to the equation above and using the differentiation property repeatedly we have

$$EI \xi^4 \hat{u}(\xi) + \kappa \hat{u}(\xi) = -mg \Rightarrow \hat{u}(\xi) = -\frac{mg}{EI} \frac{1}{\xi^4 + a^4},$$

where $a = \sqrt[4]{\kappa/EI}$. Then, in order to take the inverse transform and identify $u(x)$, we need to compute the integral

$$u(x) = -\frac{mg}{2\pi EI} \int_{-\infty}^{\infty} \frac{e^{i\xi x}}{\xi^4 + a^4} d\xi.$$  

Unfortunately, computation of this integral is by no means a trivial task. This is a common obstacle to use of the Fourier transform and/or the inverse transform. It turns out that what is needed is the residue calculus of the theory of functions of a complex variable. We do not want to get into that theory here but we can indicate the result which the residue calculus gives when applied to integrals of this type.

Let $r(\xi) = q(\xi)/p(\xi)$ be a rational function; the quotient of two polynomials in $\xi$. If we suppose that the denominator polynomial $p(\xi)$ has no zeros on the real axis (if $p(\xi)$ has real coefficients this implies the degree of $p(\xi)$ is even) and that the degree of $p(\xi)$ exceeds that of $q(\xi)$ by at least two, then the integral

$$\mathcal{I}(x) = \int_{-\infty}^{\infty} \frac{e^{i\xi x} q(\xi)}{p(\xi)} d\xi$$

exists as an improper integral. Let us denote by $\xi_k^+, \ k = 1, 2, ..., K$ and $\xi_j^-, \ j = 1, 2, ..., J$ the roots of $p(\xi)$ in the upper half plane and the lower half plane, respectively. For our treatment here we will assume these are all simple roots; thus $p'(\xi_k^+) \neq 0$, $p'(\xi_j^-) \neq 0$. When $p(\xi)$ has real coefficients $K = J$ and we can assume $\xi_j^- = \xi_k^+$. This will
A very similar computation gives 

\( \xi_i \) at the points \( \xi_1^+ \) (or \( \xi_2^- \)) is \( r_k^+ = \frac{e^{i\xi_k} q(\xi_k)}{p(\xi_k)} \) (or \( r_j^- = \frac{e^{i\xi_j} q(\xi_j^-)}{p(\xi_j^-)} \)). Then the residue theorem of complex analysis says that

\[
\frac{1}{2\pi i} \mathcal{I}(x) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{e^{i\xi} q(\xi)}{p(\xi)} d\xi = \begin{cases} \sum_{k=1}^{K} r_k^+ & x > 0, \\ -\sum_{j=1}^{J} r_j^- & x < 0. \end{cases}
\]

To apply this result to the integral of the present example we factor the denominator of the integrand:

\[
\xi_i^4 + a_4^4 = (\xi_i^2 + i a_2^2)(\xi_i^2 - i a_2^2)
\]

\[
= \left( \xi - \frac{1 + i}{\sqrt{2}} a \right) \left( \xi - \frac{1 - i}{\sqrt{2}} a \right) \left( \xi + \frac{1 + i}{\sqrt{2}} a \right) \left( \xi + \frac{1 - i}{\sqrt{2}} a \right).
\]

From this factorization we conclude that the \( \xi_i^4 + a_4^4 \) has simple zeros at the points \( \xi_1^+ = \frac{1+i}{\sqrt{2}} a \) and \( \xi_2^+ = \frac{-1+i}{\sqrt{2}} a \) in the upper half complex plane and at the points \( \xi_1^- = \frac{1-i}{\sqrt{2}} a \) and \( \xi_2^- = \frac{-1-i}{\sqrt{2}} a \) in the lower half complex plane. The derivatives of the denominator \( \xi_i^4 + a_4^4 \) at the points \( \xi_1^+ \) and \( \xi_2^+ \) are \( 4a^3 \frac{1+i}{\sqrt{2}} \) and \( 4a^3 \frac{-1+i}{\sqrt{2}} \), respectively, leading to

\[
\int_{-\infty}^{\infty} \frac{e^{i\xi} d\xi}{\xi^4 + a_4^4} = \frac{2\pi i}{4a^3} \left( \frac{e^{i\frac{(1+i)}{\sqrt{2}}} ax}{\frac{-1+i}{\sqrt{2}}} + \frac{e^{i\frac{1-1+i}{\sqrt{2}}} ax}{\frac{1+i}{\sqrt{2}}} \right)
\]

\[
= \frac{\pi i}{2a^3} \left( \frac{-1-i}{\sqrt{2}} e^{i\frac{(1+i)}{\sqrt{2}} ax} + \frac{1+i}{\sqrt{2}} e^{i\frac{-1+i}{\sqrt{2}} ax} \right)
\]

\[
= \frac{\pi e^{-1/2}}{a^3} \left( \cos \frac{ax}{\sqrt{2}} + \sin \frac{ax}{\sqrt{2}} \right), \quad x > 0.
\]

A very similar computation gives

\[
\int_{-\infty}^{\infty} \frac{e^{i\xi} d\xi}{\xi^4 + a_4^4} = \frac{\pi e^{1/2}}{a^3} \left( \cos \frac{ax}{\sqrt{2}} - \sin \frac{ax}{\sqrt{2}} \right), \quad x < 0.
\]
Going back to the expression for \( u(x) \) and simplifying the expressions involved, we have
\[
  u(x) = \begin{cases} 
    -\frac{mg}{2EIa^3} \left( \cos \frac{ax}{\sqrt{2}} + \sin \frac{ax}{\sqrt{2}} \right), & x > 0, \\
    -\frac{mg}{2EIa^3} \left( \cos \frac{ax}{\sqrt{2}} - \sin \frac{ax}{\sqrt{2}} \right), & x < 0. 
  \end{cases}
\]

We can complete the definition of \( u(x) \) by setting \( u(0) = -\frac{mg}{2EIa^3} \). Thus the (downward) vertical displacement of the loaded catwalk exhibits oscillating exponential decay toward the nominal (0) equilibrium level; it actually rises at some points as the plot below indicates.

**Figure 1**

![Typical plot of \( u(x) \)](image-url)