Separable First Order Differential Equations

Form of Separable Equations These are differential equations which take the form

\[ \frac{dy}{dx} = \frac{g(x)}{h(y)} \quad \text{or} \quad \frac{dy}{dx} = g(x)\hat{h}(y), \]

where \( g(x) \) is a continuous function of \( x \) and \( h(y) \) is a continuously differentiable function of \( y \) (to guarantee unique solutions). The two forms agree if \( \hat{h}(y) = 1/h(y) \). In differential form we can re-write such an equation as

\[ h(y) \, dy = g(x) \, dx \]

thus separating the \( y \) dependence from the \( x \) dependence. We have already encountered the simplest example, i.e., the homogeneous first order linear equation

\[ \frac{dy}{dx} + p(x) \, y = 0. \]

We can re-write this in the form

\[ \frac{1}{y} \, dy = -p(x) \, dx. \]

Method of Solution Integrating both sides of \( h(y) \, dy = g(x) \, dx \) we have

\[ \int^{y} h(r) \, dr = \int^{x} g(s) \, ds + c, \]

where \( c \) is an arbitrary constant. Then, letting \( H(y) = \int^{y} h(r) \, dr \) and \( G(x) = \int^{x} g(s) \, ds \), we have

\[ H(y) = G(x) + c \quad \text{or} \quad H(y) - G(x) - c = 0. \]

This is a parametric equation for \( y \) in terms of \( x \); if it can be solved for \( y \) to give, explicitly,

\[ y = y(x, c) = H^{-1}(G(x) + c), \]

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then we have an explicit formula for the general solution.

Example 1 Consider the differential equation
\[
\frac{dy}{dx} = xy + 2x + y + 2.
\]
Here we can factor the right hand side: \(xy + 2x + y + 2 = (x + 1)(y + 2)\). So we have
\[
\frac{1}{y + 2} \, dy = (x + 1) \, dx
\]
and, on integration, we have
\[
\log |y + 2| = \frac{(x + 1)^2}{2} + \hat{c}.
\]
Then, much as before,
\[
y + 2 = \pm \exp \left( \frac{(x + 1)^2}{2} + \hat{c} \right) = \pm e^{\hat{c}} \exp \left( \frac{(x + 1)^2}{2} \right) = c \exp \left( \frac{(x + 1)^2}{2} \right)
\]
so that the general solution becomes
\[
y(x, c) = c \exp \left( \frac{(x + 1)^2}{2} \right) - 2.
\]

Example 2 Consider the differential equation
\[
\frac{dy}{dx} = \frac{y^2}{x^2 + 3x + 2}.
\]
This becomes
\[
\frac{1}{y^2} \, dy = \frac{1}{x^2 + 3x + 2} \, dx.
\]
At this point we need to pause and provide a
Reminder: Partial Fractions Decomposition  When we need to integrate a function which is a quotient of two polynomials, i.e.,

\[ f(x) = \frac{b_1 x^{n-1} + b_2 x^{n-2} + \cdots + b_n}{a_0 x^n + a_1 x^{n-1} + \cdots a_{n-1} x + a_n}, \quad a_0 \neq 0, \]

we first need to transform \( f(x) \) into another form. This is done by the procedure of partial fractions decomposition. The first step is to factor the denominator. Assuming it has distinct roots \( r_1, r_2, \ldots, r_n \) the denominator takes the form \( a_0 (x - r_1)(x - r_2) \cdots (x - r_n) \). We then try to write

\[ f(x) = \frac{c_1}{x - r_1} + \frac{c_2}{x - r_2} + \cdots + \frac{c_n}{x - r_n}. \]

Recombining the right hand side into a single fraction with a common denominator we have

\[ f(x) = \frac{c_1 (x - r_2) \cdots (x - r_n) + \cdots + c_n (x - r_1) \cdots (x - r_{n-1})}{(x - r_1)(x - r_2) \cdots (x - r_n)}. \]

The numerator shown here, and the expression \( \frac{1}{a_0}(b_1 x^{n-1} + b_2 x^{n-2} + \cdots + b_n) \) must be the same. Equating the coefficients of corresponding powers of \( x \) gives \( n \) linear equations in the \( n \) unknowns \( c_1, c_2, \ldots, c_n \) which can be solved to obtain the desired expression \( \frac{c_1}{x - r_1} + \frac{c_2}{x - r_2} + \cdots + \frac{c_n}{x - r_n} \).

If one of the roots, say \( r_1 \), is a double root, we try instead for an expression

\[ f(x) = \frac{c_1 x + d_1}{(x - r_1)^2} + \frac{c_2}{x - r_2} + \cdots + \frac{c_{n-1}}{x - r_{n-1}}; \]

for a triple root we use \( \frac{c_1 x^2 + d_1 x + c_1}{(x - r_1)^3} \), etc.

As an example we construct the partial fractions decomposition of

\[ f(x) = \frac{x^2 - x + 2}{x^3 + 5x^2 + 8x + 4} = \frac{x^2 - x + 2}{(x + 1)(x + 2)^2}. \]
Since \( r = -2 \) is a double root, we try to achieve the form

\[
f(x) = \frac{c_1 x + d_1}{(x + 2)^2} + \frac{c_2}{x + 1}.
\]

Recombining these two fractions into a single fraction we obtain

\[
f(x) = \frac{(c_1 x + d_1)(x + 1) + c_2(x + 2)^2}{(x + 1)(x + 2)^2}
\]

\[
= \frac{(c_1 + c_2) x^2 + (c_1 + d_1 + 4c_2) x + d_1 + 4c_2}{(x + 1)(x + 2)^2}.
\]

Comparing this with the original formula for \( f(x) \) and equating coefficients of corresponding powers of \( x \) we arrive at three equations:

\[
(i) : \quad c_1 + c_2 = 1;
\]

\[
(ii) : \quad c_1 + d_1 + 4c_2 = -1;
\]

\[
(iii) : \quad d_1 + 4c_2 = 2.
\]

Substituting equation (iii) into equation (ii) we find \( c_1 = -3 \). Using this in equation (i) we obtain \( c_2 = 4 \) and then using this in equation (iii) we find \( d_1 = -14 \). Accordingly, we have

\[
f(x) = \frac{x^2 - x + 2}{x^3 + 5x^2 + 8x + 4} = \frac{-3x - 14}{(x + 2)^2} + \frac{4}{x + 1}.
\]

**Example 2 (Continued)** We return to our unfinished differential equations example. To apply partial fractions decomposition to that case we note that \( x^2 + 3x + 2 = (x + 1)(x + 2) \). So we try

\[
\frac{1}{x^2 + 3x + 2} = \frac{c_1}{x + 1} + \frac{c_2}{x + 2}
\]

\[
= \frac{c_1(x + 2) + c_2(x + 1)}{x^2 + 3x + 2} = \frac{(c_1 + c_2)x + (2c_1 + c_2)}{x^2 + 3x + 2}.
\]
Thus we need $c_1 + c_2 = 0$, $2c_1 + c_2 = 1$. This is easily solved to give $c_1 = 1$, $c_2 = -1$. Consequently we may now rewrite our differential equation in the form

$$\frac{1}{y^2} dy = \left( \frac{1}{x + 1} - \frac{1}{x + 2} \right) dx.$$ 

Integrating, we have

$$\frac{1}{y} = \log |x + 1| - \log |x + 2| + \hat{c}.$$ 

Renaming $\hat{c}$ as $\log c$, we have

$$\frac{1}{y} = \log \left( \frac{|x + 2|}{c |x + 1|} \right).$$

Again absorbing the signs into the constant $c$ we have, renaming $c$ if necessary,

$$y(x, c) = \frac{1}{\log \left( \frac{x + 2}{c(x + 1)} \right)}.$$ 

We cannot always assume that we will be able to solve the integrated equation $H(y) = G(x) + c$ to get an explicit formula for the general solution $y(x, c)$.

**Example 3** We consider the differential equation

$$\frac{dy}{dx} = \frac{\sin(x)}{\log(y)}.$$ 

Rewriting this as $\log(y) \, dy = \sin(x) \, dx$ and integrating we obtain

$$y \log(y) - y = -\cos(x) + c, \quad (y > 0)$$

which is not directly solvable in the form $y = y(x, c)$. This, of course, is one of the reasons why numerical approximation methods are as
valuable as they are. The equation shown is far from useless, however. For example, suppose we want the value of \( c \) corresponding to the initial condition \( y(0) = 2 \). Substituting \( y = 2 \) and \( x = 0 \) into the implicit formula shown we have

\[
2 \log(2) - 2 = -\cos(0) + c \implies c = 2 \log(2) - 1.
\]

The solution \( y(x) \) of the initial value problem is then the solution of the equation

\[
h(x, y) = y \log(y) - y + \cos(x) - 2 \log(2) + 1 = 0,
\]

an example of what we have called a parametric equation, and we can obtain individual values of \( y(x) \) by substituting values of \( x \) into this equation and then solving numerically, by Newton’s formula or the fixed point iteration method, for \( y \). Thus if we take \( x = .1 \), start with the value 1 obtained from the initial condition at \( y = 0 \) and apply Newton’s method

\[
y_{k+1} = y_k - \left( y_k \log(y_k) - y_k + \cos(.1) - 2 \log(2) + 1 \right) / \log(y_k)
\]

we obtain almost instantly \( y(.1) = 2.0072 \). Then with \( x = .2 \) and starting value 2.0072 we obtain \( y(.2) = 2.0285 \). We can continue in this way to build a table of values for the solution near \( x = 2 \).

**Homogeneous First Order Equations** A function \( f(x, y) \) is said to be homogeneous of degree \( n \) if

\[
f(\alpha x, \alpha y) \equiv \alpha^n f(x, y)
\]

for all values of \( x, y \) and \( \alpha \). A first order differential equation \( \frac{dy}{dx} = f(x, y) \) is said to be of homogeneous type if \( f(x, y) \) is homogeneous of degree 0, i.e.,

\[
f(\alpha x, \alpha y) \equiv \alpha^0 f(x, y) = f(x, y).
\]

for all values of \( x, y \) and \( \alpha \).
Example 4  If
\[ f(x, y) = \frac{xy + y^2}{xy} \]
then \( f(\alpha x, \alpha y) \)
\[ = \frac{\alpha x \alpha y + (\alpha y)^2}{\alpha x \alpha y} = \frac{\alpha^2(xy + y^2)}{\alpha^2(xy)} = \frac{xy + y^2}{xy} = f(x, y), \]
showing \( f(x, y) \) to be homogeneous of degree zero in this case.

The present “homogeneous” case should not be confused with the linear homogeneous case where \( f(x, y) = -p(x) y \); indeed, if it were true that
\[ -p(\alpha x) \alpha y = -p(x) y, \]
then we would have
\[ p(\alpha x) = \frac{1}{\alpha} p(x) \]
and since this must be true for all \( x \) and \( \alpha \), we would necessarily have, with \( \alpha \) replaced by \( x \) and \( x \) replaced by 1:
\[ p(x) = p(x \cdot 1) = \frac{1}{x} p(1) \]
which implies \( p(x) = p(1) \frac{1}{x} \), i.e., \( p(x) \) is a multiple of \( \frac{1}{x} \). This would correspond to the case
\[ \frac{dy}{dx} = \frac{a}{x} y \]
for which we already know that the solution is
\[ y(x, c) = c \exp(a \log |x|) = |x|^a. \]

Method of Solution  Equations of homogeneous type can be solved, at least in principle, by making use of a change of variable which reduces the equation to a separable first order equation.
Starting with \( f(x, y) \) homogeneous of degree zero in the differential equation
\[
\frac{dy}{dx} = f(x, y)
\]
we set \( y = x v \) and obtain
\[
\frac{d}{dx} x v = x \frac{dv}{dx} + v = f(x, x v) = f(1, v).
\]
Then
\[
x \frac{dv}{dx} = f(1, v) - v,
\]
or
\[
h(v) dv = \frac{1}{f(1, v) - v} dv = \frac{1}{x} dx.
\]
Integrating, we have, with \( H(v) \) an antiderivative of \( h(v) = \frac{1}{f(1, v) - v} \),
\[
H(v) = \log |x| + \hat{c} = \log |x| + \log c = \log(cx).
\]
Assuming we can find an inverse function, \( H^{-1} \), for \( H(v) \), we obtain
\[
v = H^{-1}(\log(cx))
\]
and then, since \( y = x v \), we have as the general solution
\[
y(x, c) = x H^{-1}(\log(cx)).
\]

**Example 5**  
Consider the differential equation
\[
\frac{dy}{dx} + \frac{4x + 3y}{3x + 2y} = 0.
\]
Setting \( y = x v \), we have
\[
x \frac{dv}{dx} + v = -\frac{4 + 3v}{3 + 2v}.
\]
Transposing $v$ and adding fractions we have

\[
x \frac{dv}{dx} = -\frac{4 + 3v}{3 + 2v} - v = -\frac{4 + 3v + v(3 + 2v)}{3 + 2v}
\]

\[
= -\frac{2v^2 + 6v + 4}{3 + 2v} \Rightarrow x \frac{dv}{dx} = -2 \frac{(v + 2)(v + 1)}{3 + 2v}.
\]

Changing to “differential form” this is

\[
\frac{3 + 2v}{(v + 2)(v + 1)} dv = -\frac{2}{x} dx.
\]

Then, applying the method of partial fractions,

\[
\left(\frac{1}{v + 2} + \frac{1}{v + 1}\right) dv = -\frac{2}{x} dx.
\]

Integrating, we have

\[
\log |v + 2| + \log |v + 1| = -2 \log |x| + \hat{c}.
\]

This gives

\[
\log |(v + 2)(v + 1)| = -2 \log |cx| = \log \frac{1}{c^2 x^2}.
\]

Thus we have

\[
(v + 2)(v + 1) = \pm \frac{1}{c^2 x^2}.
\]

Letting $-a = \pm \frac{1}{c^2}$ be a constant of arbitrary sign, we have

\[
v^2 + 3v + 2 + \frac{a}{x^2} = 0.
\]

Solving this quadratic equation for $v$ in terms of $x$ and $a$,

\[
v(x, a) = \frac{1}{2} \left( -3 \pm \sqrt{9 - 4(2 + \frac{a}{x^2})} \right)
\]
Remembering that $y = x v$, in terms of the original dependent variable $y$ the solution is

$$y(x, a) = \frac{x}{2} \left(-3 \pm \sqrt{1 - \frac{4a}{x^2}}\right).$$

However, it would be misleading to give the impression that matters always work out so that the solution can be obtained in closed form. In fact this is often not the case as we see in the next example.

**Example 6**  We consider the very simple differential equation

$$\frac{dy}{dx} = \frac{x + y}{x - y}.$$ 

Setting $y = x v$ we have

$$x \frac{dv}{dx} + v = \frac{x + x v}{x - x v} = \frac{1 + v}{1 - v}$$

or

$$x \frac{dv}{dx} = \frac{1 + v}{1 - v} - v = \frac{1 + v - v(1 - v)}{1 - v} = \frac{1 + v^2}{1 - v}.$$ 

This is the same as

$$\frac{1 - v}{1 + v^2} \frac{dv}{x} = \frac{1}{x} dx$$

or

$$\frac{1}{1 + v^2} dv - \frac{v}{1 + v^2} dv = \frac{1}{x} dx.$$ 

Then, integrating, we have

$$\tan^{-1} v - \frac{1}{2} \log(1 + v^2) = \log |x| + c.$$ 

In this case there is essentially no hope of solving for $v$ to obtained $v = v(x, c)$ with $v(x, c)$ an expression in terms of elementary functions; the best we can do is to say that the general solution $y(x, c) = x v(x, c)$ satisfies the parametric equation

$$\tan^{-1} \frac{y}{x} - \frac{1}{2} \log \left(1 + \frac{y^2}{x^2}\right) = \log |x| + c.$$
QuickCheck Exercises

1. Find the solution of the initial value problem

\[ \frac{dy}{dx} = (1 + y^2)(x - \pi), \quad y \left( \frac{3\pi}{4} \right) = 1. \]

2. Find the general solution of

\[ \frac{dy}{dx} = \frac{x^2 + y^2}{xy + x^2} \]

as the solution of a parametric equation \( h(x, y, c) = 0 \). Then find the value of \( c \) corresponding to the initial condition \( y(1) = 2 \). Find a four decimal place approximation to \( y(1.2) \).

3. Find the solution of

\[ \frac{dy}{dx} = \frac{y^2 - 1}{2yx^2} \]

satisfying the initial condition \( y(1) = 3 \).

4. The velocity, \( v(t) \), of a certain mass sliding along a surface with friction satisfies

\[ \frac{dv}{dt} + v + v^2 = 0. \]

If the initial velocity is \( v(0) = 10 \) meters per second, when is the velocity reduced to 1 meter per second?