Second Order Differential Equations

Equivalent Forms; Initial Value Problems  In general, second order differential equations take the form

$$\frac{d^2y}{dx^2} = f \left(x, y, \frac{dy}{dx}\right),$$

where $f(x, y, z)$ is a continuous function of the variables $x$, $y$ and $z$ in some region $\mathcal{R} \subset \mathbb{R}^3$ and is continuously differentiable with respect to $y$ and $z$ throughout that region.

If we define $z = \frac{dy}{dx}$, then we can write the above equation equivalently as a system of two (coupled) first order equations:

$$\frac{dy}{dx} = z; \quad \frac{dz}{dx} = f(x, y, z).$$

Written in this form and drawing on experience with first order equations, we would expect to have to give initial values

$$y(x_0) = y_0, \quad z(x_0) = z_0,$$

in order to specify an initial value problem. Going back to the second order equation, this means we need to give

$$y(x_0) = y_0, \quad \frac{dy}{dx}(x_0) = z_0; \quad \text{equivalently:} \quad \frac{dy}{dx}(x_0) = y_1,$$

in order to specify an initial value problem for it.

General Solutions  We say that an expression $y(x, c_1, c_2)$ is a general solution for the second order differential equation above if, as a function of $x$, it satisfies the differential equation, and, given initial values as above at a point $x_0$, with $(x_0, y_0, y_1)$ in
the region R where \( f(x, y, z) \) has the required properties, we can solve the equations

\[
y(x_0, c_1, c_2) = y_0, \quad \frac{dy}{dx}(x_0, c_1, c_2) = y_1
\]

for \( c_1 \) and \( c_2 \) to satisfy that initial value problem.

**Example 1** Consider the second order differential equation

\[
\frac{d^2 y}{dx^2} = \frac{2}{x} \frac{dy}{dx} - \frac{2}{x^2} y
\]

in the region \( x > 0 \), and let us try to find solutions of the form \( y = x^r \). Substituting, we have

\[
r(r - 1) x^{r-2} = 2r x^{r-2} - 2x^{r-2}
\]

or

\[
x^{r-2} (r^2 - 3r + 2) = 0.
\]

Since \( x^{r-2} \) is not identically zero, we need to solve the so-called **indicial equation**

\[
r^2 - 3r + 2 = 0 \quad \rightarrow \quad r = 2 \text{ or } r = 1.
\]

So we conclude that \( y(x) = x \) and \( y(x) = x^2 \) are solutions, which is easily checked out. Since the equation is linear with respect to \( y \) (if \( y \) and \( \hat{y} \) are solutions, so is any linear combination of the two) it then makes sense to try a general solution of the form

\[
y(x, c_1, c_2) = c_1 x + c_2 x^2.
\]

To satisfy initial conditions as specified above at \( x_0 > 0 \) we need

\[
c_1 x_0 + c_2 x_0^2 = y_0
\]

\[
c_1 + 2c_2 x_0 = y_1,
\]

a system of two linear algebraic equations. The determinant of the coefficients of \( c_1 \) and \( c_2 \) on the left hand side is \( 2x_0^2 - x_0^2 = x_0^2 \neq 0 \) for \( x_0 \neq 0 \) so we conclude we can always obtain a unique
solution of these equations if \( x_0 \neq 0 \). We conclude therefore that
\( y(x, c_1, c_2) = c_1 x + c_2 x^2 \) is the general solution of the second
order differential equation \( \frac{d^2 y}{dx^2} = \frac{2}{x} \frac{dy}{dx} - \frac{2}{x^2} y \) for \( x \neq 0 \).

A General Method of Solution in the “Autonomous” Case

If our second order differential equation does not depend explicitly on \( x \), which is technically referred to as the autonomous, i.e., ”self-governing”, case, we can employ a transformation of
variables to reduce the second order equation to two first order
differential equations. Assuming the equation is
\[
\frac{d^2 y}{dx^2} = f(y, \frac{dy}{dx}),
\]

we let
\[
\frac{dy}{dx} = z.
\]
Substituting this into the second derivative we have
\[
\frac{d^2 y}{dx^2} = \frac{dz}{dx} = \frac{dz}{dy} \frac{dy}{dx} = z \frac{dz}{dy}
\]
and thus the original second order equation becomes
\[
z \frac{dz}{dy} = f(y, z).
\]
If we can solve this last equation to get \( z = z(y, c_1) \), then the
equation used to define \( z \) becomes
\[
\frac{dy}{dx} = z(y, c_1)
\]
which, being in separable form
\[
\frac{1}{z(y, c_1)} dy = dx
\]
can be integrated to give
\[
\int^y \frac{1}{z(\eta, c_1)} d\eta = x + c_2.
\]
Then, at least in principle, we can solve this to get \( y = y(x, c_1, c_2) \). Success of the project ultimately depends on being able to get a closed form solution for the transformed second order equation, \( z \frac{dz}{dy} = f(y, z) \), which has now become a first order equation in \( z \) and \( y \).

**Example 2** We consider the autonomous second order differential equation

\[
\frac{d^2y}{dx^2} = y \frac{dy}{dx}.
\]

Following the plan indicated above, we arrive at

\[
z \frac{dz}{dy} = yz \text{ or } \frac{dz}{dy} = y.
\]

This is solved immediately to obtain

\[
z = z(y, c_1) = \frac{y^2 + c_1}{2}.
\]

Then we have

\[
\frac{dy}{dx} = \frac{y^2 + c_1}{2}.
\]

There are three cases here, depending on whether \( c_1 \) is 0, positive, or negative. Taking just the case where \( c_1 \) is positive, we write \( c_1 = c^2 \) and we have

\[
\frac{2}{c^2 + y^2} dy = dx.
\]

Setting \( y = cw \) we have

\[
\frac{2}{c} \left( \frac{1}{1 + w^2} \right) dw = dx
\]

and we integrate to obtain

\[
\frac{2}{c} \tan^{-1}(w) = x + c_2.
\]
Then \( w = \tan\left(\frac{c}{2}(x + c_2)\right) \) and we have as the general solution

\[
y = y(x, c, c_2) = c \tan\left(\frac{c}{2}(x + c_2)\right).
\]

The cases corresponding to \( c_1 = 0 \) and \( c_1 < 0 \) are handled in a similar way but different expressions are obtained. (Try those cases as an exercise.)