Hwk 1 - Math 3224 - Summer 1, 2004

(1) $x^2-1=(x+1)(x-1) \Rightarrow A =]-\infty,-1] \cup [1,+\infty[$

$x^2-x-6=0 \Rightarrow B=\{3,-2\} \subset A$ so $\boxed{A\cup B = A}$ $\boxed{A\cap B = B}$

(2) $B=\{0,-1,1\}$ $A=\{-1,2,1,-2,4,3,0\}$ $\boxed{\forall x\in B,\ x\in A \text{ so } B\subset A}$

(2-a) $(\exists x\in A)(x=4)(x\notin B)$ $\boxed{\text{so } B \not\supset A}$

(2-b) $\boxed{C = A\setminus B = \{2,-2,4,3\}}$

(3-a) $f(x)=\sqrt{2-x^2}$ $\boxed{\begin{array}{l} Dom(f)=\{x\in\mathbb{R};\ 2-x^2\geq 0\} = [-\sqrt{2},\sqrt{2}] \\ Ran(f)=[0,\sqrt{2}] \end{array}}$

(3-b) $|x-2|\leq 1 \Leftrightarrow 1\leq x\leq 3 \Leftrightarrow x\in[1,3]$

$A=[1,3]\cap\left]-\frac{3}{2},\frac{5}{2}\right[= \boxed{\left[1,\frac{5}{2}\right[= A}$

(4-a) $-1\leq \cos n\leq 1 \Rightarrow 2\leq 3+\cos n\leq 4 \Rightarrow \boxed{\frac{1}{4}\leq a_n = \frac{1}{3+\cos n}\leq\frac{1}{2}}$

(4-b) $a_n=b_n c_n$; $b_n = 3-\frac{2}{n}$; $c_n=2+\frac{5}{n}$

$\lim_{n\to\infty} b_n = 3$ $\lim_{n\to\infty} c_n = 2$ so apply $\lim[b_n c_n] = \lim b_n \cdot \lim c_n$

so $\boxed{\lim_{n\to\infty} a_n = 6}$

(5) $a_n = \frac{\sin(1000n)}{n}$; $|a_n|\leq\frac{1}{n}$ so

$(\forall\varepsilon>0)(\exists N\in\mathbb{N})(N\geq\frac{1}{\varepsilon})(n\geq N\Rightarrow |a_n|\leq\varepsilon)$

$b_n=\frac{1}{n}+\frac{\cos n}{1+n} = c_n+d_n$; $c_n=\frac{1}{n}$; $d_n=\frac{\cos n}{1+n}$

$|b_n|\leq\frac{2}{n}$ so $(\forall\varepsilon>0)(\exists N\in\mathbb{N})(N\geq\frac{2}{\varepsilon})(n\geq N\Rightarrow |b_n|\leq\varepsilon)$

Hwk2 - Math3884 - Summer 1, 2004

(1) $A-B=\{h,b\}$; $B-A=\{e,d\}$ $\quad A-B\neq B-A$

(2) $A\cap(B\cup C)\overset{?}{=}(A\cap B)\cup(A\cap C)$

(*) $x\in A\cap(B\cup C)\Rightarrow\left\{\begin{matrix}x\in A\\ \&\\ x\in B\cup C\end{matrix}\right.\rightarrow\left.\begin{matrix}x\in A\ \&\ x\in B\\ \text{or}\\ x\in A\ \&\ x\in C\end{matrix}\right\}\Rightarrow x\in(A\cap B)\cup(A\cap C)$

so $A\cap(B\cup C)\subset(A\cap B)\cup(A\cap C)$

(**) $x\in(A\cap B)\cup(A\cap C)\Rightarrow\left.\begin{matrix}x\in A\ \&\ x\in B\\ \text{or}\\ x\in A\ \&\ x\in C\end{matrix}\right\}\Rightarrow\left.\begin{matrix}x\in A\\ \&\\ (x\in B)\text{ or }(x\in C)\end{matrix}\right\}\Rightarrow x\in A\cap(B\cup C)$

so $(A\cap B)\cup(A\cap C)\subset A\cap(B\cup C)$

(3) $n=1$ $P(1)$ $1+\frac{1}{2}=2-\frac{1}{2}$

(a) Assume $P(n)$ $1+\frac{1}{2}+\frac{1}{4}+\dots+\frac{1}{2^n}=2-\frac{1}{2^n}$

$P(n+1)$? $1+\frac{1}{2}+\frac{1}{4}+\dots+\frac{1}{2^n}+\frac{1}{2^{n+1}}=2-\frac{1}{2^n}+\frac{1}{2^{n+1}}=2-\left[\frac{1}{2^n}-\frac{1}{2^{n+1}}\right]=2-\frac{2-1}{2^{n+1}}=\frac{2-1}{2^{n+1}}$

so $P(n+1)$

(b) $n=3$ $P(3)$ $6+1\leq 2^3=8$ $\quad P(3)$

Assume $P(n)$ $2n+1\leq 2^n$. $\quad 2(n+1)+1=2n+1+2\leq 2^n+2\leq 2^n+2^n$
$\leq 2.2^n=2^{n+1}\Rightarrow P(n+1)$

(4-1) $0<x<y\Rightarrow 1<x^{-1}y\Rightarrow y^{-1}<x^{-1}$ i.e $\frac{1}{y}<\frac{1}{x}$

(4-2) $x^2=x.x<x\rightarrow x^{-1}(x.x)<x^{-1}.x=1\Rightarrow(x^{-1}.x)x<1\Rightarrow x<1$

(5) $Dom(f)=]0,+\infty[$ $\quad Dom(g)=\mathbb{R}$; $Ran(f)=]0,+\infty[$; $Ran(g)=\mathbb{R}^+=[0,+\infty[$

$\lim_{x\to0^+}x\ln x=\lim_{y\to+\infty}\frac{1}{y}\ln\frac{1}{y}=\lim_{y\to\infty}-\frac{1}{y}=0$ (l'Hospital Rule)

$f\circ g(x)=f[x^2]=x^2\ln(x^2)$; $g\circ f(x)=g[x\ln x]=x^2(\ln x)^2$

$f\circ g\neq g\circ f$

Mid Term exam summer 1 - 2004

(1) $|a_m^2 - a_n^2| = |a_n + a_m||a_n - a_m| \le 2M|a_n - a_m|$

$(\forall \varepsilon > 0)(\exists N \in \mathbb{N})\left(m \ge n \ge N \Rightarrow |a_n - a_m| \le \frac{\varepsilon}{2M}\right)$ then $|a_m^2 - a_n^2| \le \varepsilon$

(2) $a_n = \sqrt{n+1} - \sqrt{n} = \frac{1}{\sqrt{n+1}+\sqrt{n}} \to 0 \qquad |a_n| \le \frac{1}{2\sqrt{n}} \qquad \frac{1}{2\sqrt{n}} \le \varepsilon \Rightarrow \sqrt{n} \ge \frac{1}{2\varepsilon} \Rightarrow n \ge \frac{1}{4\varepsilon^2}$

$(\forall \varepsilon > 0)(\exists N \in \mathbb{N})\left(n \ge N \Rightarrow |a_n| \le \frac{1}{2\sqrt{n}} \le \varepsilon\right)$

$N = \text{int} \ge \frac{1}{4\varepsilon^2}$

(3) $a_n = \frac{n+1}{3n-2} \quad a_m - a_n = \frac{m+1}{3m-2} - \frac{n+1}{3n-2} \Rightarrow \frac{(m+1)(3n-2) - (n+1)(3m-2)}{(3m-2)(3n-2)} = \frac{\cancel{3mn} - 2m + 3n - 2 - [\cancel{3mn} - 2n + 3m - 2]}{(3m-2)(3n-2)}$

$a_m - a_n = \frac{-5m + 5n}{(3m-2)(3n-2)} = -5\frac{m}{(3m-2)(3n-2)} + 5\frac{n}{(3m-2)(3n-2)}$

$|a_n - a_m| \le \frac{5}{m} + \frac{5}{n} \quad ; \quad \frac{5}{m} \le \frac{\varepsilon}{2} \Rightarrow m \ge \frac{10}{\varepsilon}$ Then $\left(N = \text{int} \ge \frac{10}{\varepsilon}\right) \begin{matrix} m \ge N \\ n \ge N \end{matrix} \Rightarrow |a_m - a_n| \le \varepsilon$

$\lim a_n = \frac{1}{3} \qquad \left|a_n - \frac{1}{3}\right| = \left|\frac{3n+3 - 3n+2}{3(3n-2)}\right| = \frac{5}{3}\left|\frac{1}{3n-2}\right| \sim \frac{5}{3}\frac{1}{3n} \le \varepsilon$ when $\frac{5}{3}\frac{1}{n} \le \varepsilon \Rightarrow (N)$

(4) $f : [0,9] \to \mathbb{R} \qquad \bar{x}$

$|\sqrt{x} - \sqrt{\bar{x}}| = \left|\frac{x - \bar{x}}{\sqrt{x} + \sqrt{3}}\right| \le \frac{1}{\sqrt{3}}|x - \bar{x}| \le \varepsilon \Rightarrow \boxed{\delta = \sqrt{3}.\varepsilon}$ uniformly continuous.

$\text{dom}(f) = [0,9] \quad \text{Ran} f = [0,3]$

(5) $x = 2 \qquad \lim_{x \to 2^-} f(x) = 2 \; ; \; \lim_{x \to 2^+} f(x) = 4 - 2 = 2$ so f is continuous

$\{a_n\}$ & $\{b_n\} \qquad a_n = 3 - \frac{1}{n} \quad b_n = 3 + \frac{1}{n}$

$\lim_{n \to +\infty} f\left(3 - \frac{1}{n}\right) = \lim_{n \to \infty} 4 - 3 + \frac{1}{n} = 1 \qquad \lim_{n \to \infty} f\left(3 + \frac{1}{n}\right) = \frac{1}{2}\left[3 + \frac{1}{n}\right] = \left(\frac{3}{2} + \frac{1}{2n}\right) = \frac{3}{2}$

so f is discontinuous at $x = 3$.