REFERENCE SEQUENCES - Abstract- Math324- KONATE

- **Definition:** a sequence is an infinite list of numbers
- **A reference list of limits**

1. $\lim_{n \to +\infty} \frac{\ln(n)}{n} = 0$
2. $\lim_{n \to +\infty} n^{1/n} = 1$
3. $\lim_{n \to +\infty} x^{1/n} = 1 \quad x > 0\,;\, x > 0$
4. $\lim_{n \to +\infty} x^n = 0 \quad |x| < 1$
5. $\lim_{n \to +\infty} (1 + \frac{x}{n})^n = \mathrm{e}^x$
6. $\lim_{n \to +\infty} \frac{x^n}{n!} = 0$

- **Two useful inequalities**

For n large, we have both:

$n! > 2^n$

$n! > e^n$.

- **One useful tool**

L'Hospital Rule:

If a_n is such that: $a_n = \dfrac{f(n)}{g(n)}$ and

$$\begin{cases} \lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} g(x) = 0 \\ or \\ \lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} g(x) = +\infty \end{cases}$$

then

$$\lim_{n \to +\infty} a_n = \lim_{n \to +\infty} \frac{f'(n)}{g'(n)}.$$

- **Hints**

If a sequence contains $\ln(n)$ or e^n then think of using L'Hospital Rule to calculate the limit when $n \to +\infty$.

SERIES - Abstract- Math3224- KONATE

- **Definition:** an infinite series $\sum_{n=1}^{+\infty} a_n$ is an infinite sum of numbers.
- **A reference list of series**
- • **reference convergent series**

1. $\sum_{n=0}^{+\infty} ar^n$ when $|r| < 1$ then

$$\sum_{n=0}^{+\infty} ar^n = \frac{a}{1-r} \quad \text{and} \quad \sum_{n=1}^{+\infty} ar^n = \frac{ar}{1-r}$$

2. $\sum_{n=1}^{+\infty} \frac{1}{n!}$

3. $\sum_{n=1}^{+\infty} \frac{1}{n^p}$ when $p > 1$

4. $\sum_{n=1}^{+\infty} \frac{1}{n(n+1)} = 1$

- • **reference divergent series**

1. $\sum_{n=0}^{+\infty} ar^n$ when $|r| \geq 1$

2. $\sum_{n=1}^{+\infty} \frac{1}{n}$

3. $\sum_{n=1}^{+\infty} a_n$ when $\lim_{n \to +\infty} a_n$ DNE or $\lim_{n \to +\infty} a_n \neq 0$

4. $\sum_{n=1}^{+\infty} \frac{1}{n^p}$ when $p \leq 1$

- **Tips on how to handle the study of a series**
- • **case1: no term within the series $\sum_{n=1}^{+\infty} a_n$ is negative**

1. *If $\lim_{n \to +\infty} a_n$ is different from zero then the series is divergent.*
2. *If this limit is null then go for further investigations. Make use of one of the following Tests:(the n-th tern of the given series is a_n and that of a reference series is b_n)*

2.1 Direct Comparison Test

- if $a_n \leq b_n$ and $\sum_{n=1}^{+\infty} b_n$ is convergent then so is $\sum_{n=1}^{+\infty} a_n$

- if $a_n \geq b_n$ and $\sum_{n=1}^{+\infty} b_n$ is divergent then so is $\sum_{n=1}^{+\infty} a_n$

2.2 Limit Comparison Test

- if $\lim_{n \to +\infty} \frac{a_n}{b_n} = C > 0$ then $\sum_{n=1}^{+\infty} a_n$ and $\sum_{n=1}^{+\infty} b_n$ are of the same type.

- if $\lim_{n \to +\infty} \frac{a_n}{b_n} = 0$ and $\sum_{n=1}^{+\infty} b_n$ converges then $\sum_{n=1}^{+\infty} a_n$ converges too;

- if $\lim_{n \to +\infty} \frac{a_n}{b_n} = +\infty$ and $\sum_{n=1}^{+\infty} b_n$ diverges then $\sum_{n=1}^{+\infty} a_n$ diverges too.

2.3 Ratio Test

Set $\lim_{n \to +\infty} \frac{a_{n+1}}{a_n} = \rho$. We have:

- if $\rho < 1$ then $\sum_{n=1}^{+\infty} a_n$ converges ;

- if $\rho > 1$ or $\rho = +\infty$ then $\sum_{n=1}^{+\infty} a_n$ diverges ;

- if $\rho = 1$ then this test is inconclusive .

2.4 Integral Test

This test is particularly usefull when it comes to evaluate the limit of some specific series.

It says:

Assume $a_n = f(n)$ where f is a positive, decreasing function over $[1, +\infty[$.

- if $\int_1^{+\infty} f(x)dx$ is convergent then so is $\sum_{n=1}^{+\infty} a_n$,

- if $\int_1^{+\infty} f(x)dx$ is divergent then so is $\sum_{n=1}^{+\infty} a_n$.

- When $\int_1^{+\infty} f(x)dx$ and $\sum_{n=1}^{+\infty} a_n$ are convergent then the n-th partial sum $S_n = \int_1^{n} f(x)dx$ is an approximation to $S = \sum_{n=1}^{+\infty} a_n$ and the error E_n we are committing in replacing $S = \sum_{n=1}^{+\infty} a_n$ by S_n is such that:

$\int_{n+1}^{+\infty} f(x)dx \leq S - S_n \leq \int_n^{+\infty} f(x)dx.$

• • **case1: Alternating series** $\sum_{n=1}^{+\infty} (-1)^{n+1} a_n \, ; \, a_n > 0$

if an alternating series $\sum_{n=1}^{+\infty} (-1)^{n+1} a_n \, ; \; a_n > 0$ is such that

$$\begin{cases} a_n \geq a_{n+1} \\ \lim\limits_{n \to +\infty} a_n = 0 \end{cases}$$

then it is convergent and:

$|S - S_n| \leq a_{n+1}.$

Plus, S_n is an overestimation of S if $(-1)^{n+1} = -1$ and an underestimation if $(-1)^{n+1} = 1$.

NOTICE: *If you change the order of the terms in a convergent alternating series, it may become divergent. An alternating series which converges independently of the order of its ternms is said to be absolutely convergent.*
$\sum_{n=1}^{+\infty} (-1)^{n+1} a_n \, ; \, a_n > 0$ is absolutely convergent if $\sum_{n=1}^{+\infty} |a_n|$ is convergent.