Numerical Methods – Abstract

(*) Consider :
$$(E1)\begin{cases} y''(t) + a(t)\,y'(t) + b(t)\,y(t) = g(t) \\ y(0) = y_0 \quad ; \quad y'(0) = y_1 \end{cases}$$

(1) Convert (E1) into a system of 2 first order differential equation:

set $\begin{cases} x_1(t) = y(t) \\ x_2(t) = y'(t) \end{cases} \rightarrow \begin{cases} x_1'(t) = 0x_1 + x_2 \\ x_2'(t) = -b(t)x_1 - a(t)x_2 + g(t) \end{cases}$

set $X(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$ then

$$\begin{cases} X'(t) = F(t, X) \\ X(0) = \begin{pmatrix} y_0 \\ y_1 \end{pmatrix} \end{cases} \quad \text{with} \quad F(t, X) = \begin{pmatrix} 0 & 1 \\ -b(t) & -a(t) \end{pmatrix} X + \begin{pmatrix} 0 \\ g(t) \end{pmatrix}$$

(2) discretize an interval $I = [0, T]$ with a mesh $h = \frac{T}{N}$ ~~$\frac{1}{10}$, $\frac{1}{100}$~~
usually h is taken to be $\frac{1}{10}$; $\frac{1}{20}$; $\frac{1}{50}$; $\frac{1}{100}$ etc...
to get the mesh points: $t_0 = 0 \quad t_1 = t_0 + h, \ldots \quad t_{i+1} = t_i + h = t_0 + (i+1)h$

(3) Euler Method: start from $X_0 = \begin{pmatrix} y_0 \\ y_1 \end{pmatrix}$ and when X_i is known
Compute : $X_{i+1} = X_i + h\,F(t_i, X_i)$

(4) Runge Kutta Method: start from $X_0 = \begin{pmatrix} y_0 \\ y_1 \end{pmatrix}$ when X_i is known
then Compute :
$$K_1 = F(t_i, X_i)$$
$$K_2 = F\left(t_i + \frac{h}{2}, X_i + \frac{h}{2}K_1\right)$$
$$K_3 = F\left(t_i + \frac{h}{2}, X_i + \frac{h}{2}K_2\right)$$
$$K_4 = F(t_{i+1}, X_i + hK_3) \quad \text{and}$$

$$X_{i+1} = X_i + \frac{h}{6}\left[K_1 + 2K_2 + 2K_3 + K_4\right]$$

Observation: the numerical methods provide you ~~both~~ with both an an approximation of $y(t)$ and $y'(t)$.