Examples of ε - δ proofs:

1) Prove that $\lim_{x\to 2} 5x+3=13$

<u>Proof</u>:

Let $\varepsilon > 0$ be given.

a) Guessing a value for δ:

Given $\varepsilon > 0$ find a $\delta > 0$ such that

$$0<|x-2|<\delta \Rightarrow |(5x+3)-13|<\varepsilon$$
$$\Rightarrow |5x-10|<\varepsilon$$
$$\Rightarrow 5|x-2|<\varepsilon$$
$$\Rightarrow |x-2|<\varepsilon/5$$

This suggests that we choose $\delta = \varepsilon/5$.

b) Showing that this δ works.

Given $\varepsilon > 0$, choose $\delta = \varepsilon/5$.

Then $0<|x-2|<\delta \Rightarrow |(5x+3)-13| = |5x-10|$

$$= 5|x-2|$$
$$< 5\delta$$
$$= 5(\frac{\varepsilon}{5})$$
$$= \varepsilon.$$

Thus $|(5x+3)-13|<\varepsilon$ whenever $0<|x-2|<\delta$

So by the definition of a limit, $\lim_{x\to 2} 5x+3=13$.

2) Consider the function f(x) = -2x+5. How close to a =1 must we hold x to make sure that f(x) lies within 1.5 units of f(a) = 3?

<u>Solution</u>:

What we want to know is: when is $|f(x)-L|<1.5$?

$$|f(x)-3|<1.5 \Rightarrow |(-2x+5)-3|<1.5$$
$$\Rightarrow |-2x+2|<1.5$$
$$\Rightarrow -1.5<-2x+2<1.5$$
$$\Rightarrow -1.5-2<-2x+2-2<1.5-2$$
$$\Rightarrow -3.5<-2x<-0.5$$
$$\Rightarrow -1.75<-x<-0.25$$
$$\Rightarrow 0.25<x<1.75$$
$$\Rightarrow 0.25-1<x-1<1.75-1$$
$$\Rightarrow -0.75<x-1<0.75$$
$$\Rightarrow |x-1|<0.75$$

So we choose δ=0.75 for the given $\varepsilon = 1.5$, and we are done!

3) Suppose that $\lim_{x \to a} f(x) = \mathrm{L}$ and $\lim_{x \to a} g(x) = \mathrm{M}$.

Prove that $\lim_{x \to a}(f(x) + g(x)) = \mathrm{L} + \mathrm{M}$ (Sum Rule).

<u>Proof</u>:

Let $\varepsilon > 0$ be given.

$\exists\, \delta_1 > 0$ such that $0 < |x - a| < \delta_1 \Rightarrow |f(x) - \mathrm{L}| < \varepsilon / 2$, and

$\exists\, \delta_2 > 0$ such that $0 < |x - a| < \delta_2 \Rightarrow |g(x) - \mathrm{M}| < \varepsilon / 2$.

We need to find a $\delta > 0$ such that, $0 < |x - a| < \delta \Rightarrow |(f(x) + g(x)) - (\mathrm{L} + \mathrm{M})| < \varepsilon$.

$$\begin{aligned}|(f(x) + g(x)) - (\mathrm{L} + \mathrm{M})| &= |f(x) - \mathrm{L} + g(x) - \mathrm{M}| \\ &\le |f(x) - \mathrm{L}| + |g(x) - \mathrm{M}| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \qquad \text{,whenever } |x - a| < \delta_1 \text{ and } |x - a| < \delta_2.\end{aligned}$$

So if we take $\delta = \min\{\delta_1, \delta_2\}$, then $0 < |x - a| < \delta$ will always ensure that $|(f(x) + g(x)) - (\mathrm{L} + \mathrm{M})| < \varepsilon$.