

Bounded Error Parameter Estimation for Models Described by Ordinary and Delay Differential Equations

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ABSTRACT

In this paper we focus on the problem of parameter identification for non-linear dynamical systems in the case when the number of data samples are too small for standard statistical analysis. The models are described by ordinary and delay differential equations with bounded errors. When the number of data samples is very small, standard validation methods are not applicable because classical statistical asymptotic theory relies on the behavior of the estimated parameter as the number of samples grows large. We present a new computational method that can be used to for solving this problem for a specific class of models. Although the assumptions lead to a restricted class of models, the new algorithm is computationally efficient for this class of problems. We introduce the basic ideas, provide some theoretical results needed for the convergence of the method and present numerical examples to illustrate the approach.

I. INTRODUCTION

The overall goal of this effort is to improve the modeling process for problems where only a small amount of data can be collected. Since the data sets are limited, we need to make the assumption that the random variable that represents error in the statistical model has bounded output. Our primary goal is to design an informative experiment in order to identify the unknown model parameters, to estimate the parameters, and finally to assign a measure of quality to that estimate.

Clearly one can use parameter estimation algorithms like least squares or maximum likelihood to estimate the unknown parameters, given a small data set, but one can not use the the corresponding statistical theory to validate the estimate. To validate a parameter estimated from a small data set, we use bounded error parameter identification. This validation technique maps sets in the model output to corresponding sets in the parameter space. Using the dynamics of the model, bounded error parameter identification can assign a measure of quality to the estimated parameter when traditional statistical techniques cannot. In this paper we will introduce a new

bounded error identification procedure that is particularly applicable to the design of experiments problem.

A. The Mathematical Model

We focus on mathematical models of the form

$$\dot{x}(t, \theta) = g(t, x(t, \theta), \theta), \quad (1)$$

$$x(0, \theta) = x_0, \quad (2)$$

with observation given by

$$y(t, \theta) = C(x(t, \theta)). \quad (3)$$

Here, $\theta \in \Theta \subset \mathbb{R}^p$ is the unknown parameter, $t \in [0, T]$, $x : [0, T] \times \Theta \rightarrow X \subset \mathbb{R}^n$ is the state vector, $g : [0, T] \times X \times \Theta \rightarrow X$ is continuously differentiable, $C : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is an operator, and $y \in Y \subset \mathbb{R}^m$ is the vector of observations.

B. The Statistical Model

Since a mathematical model almost never exactly fits experimental data, we consider a statistical model that takes this error into account. The statistical model is

$$D_j = y(t_j, \theta_0) + \mathcal{E}_j \quad (4)$$

where $\mathcal{E}_j = [\mathcal{E}_j^1, \dots, \mathcal{E}_j^m]^T$ is a random variable that represents noise in the data collection process. We assume that the \mathcal{E}_j 's are independent with $E[\mathcal{E}_j] = 0$ and $P(a_j^i \leq \mathcal{E}_j^i \leq b_j^i) = 1$ for finite a_j^i, b_j^i . The important features of the statistical model are that the random variables that represent error have an expected value of 0 and bounded range. The assumption that $E[\mathcal{E}_j] = 0$ is an assumption that the mathematical model is "good". That is to say, $E[D_j] = y(t, \theta_0)$. Note that it is not necessary for the theorems stated in Section II. The bounded range will let us say something about the quality of a parameter estimate or the quality of the design of an experiment.

C. Sensitivity Equations

Accurately solving for the sensitivities of the model with respect to the unknown parameters is one of the most important parts of both parameter estimation and parameter validation. In the parameter estimation process we will use gradient based algorithms that require numerous calculations of the sensitivities of the model and in the bounded error parameter identification problem we also require information about these sensitivities to use the theory. Additionally, all of the procedures for designing experiments require the sensitivity of the model. The sensitivity of the model with respect to the parameter is defined to be

$$\begin{aligned} s_{\theta_1}(t, \theta) &\equiv \left[\frac{\partial y_1(t, \theta)}{\partial \theta_1} \quad \frac{\partial y_2(t, \theta)}{\partial \theta_1} \quad \dots \quad \frac{\partial y_m(t, \theta)}{\partial \theta_1} \right]_T \\ s_{\theta_2}(t, \theta) &\equiv \left[\frac{\partial y_1(t, \theta)}{\partial \theta_2} \quad \frac{\partial y_2(t, \theta)}{\partial \theta_2} \quad \dots \quad \frac{\partial y_m(t, \theta)}{\partial \theta_2} \right]_T \\ &\vdots \\ s_{\theta_p}(t, \theta) &\equiv \left[\frac{\partial y_1(t, \theta)}{\partial \theta_p} \quad \frac{\partial y_2(t, \theta)}{\partial \theta_p} \quad \dots \quad \frac{\partial y_m(t, \theta)}{\partial \theta_p} \right]_T \end{aligned}$$

When the system is described by an exact model, obtaining sensitivities can be straight forward. However, more care has to be taken when solving for the sensitivities of a model described by a differential equation. In this paper we use the continuous sensitivity method described in [1] and [2].

D. Numerical Approximations

We say a partition of the set A is μ -fine if for each $a \in A$ there exists a $p \in \rho^\mu$ such that $\|a - p\|_2 < \mu$. Suppose that $\rho^\mu = [p_1, \dots, p_M]$ is a μ -fine partition of $[0, T]$. We will solve the sensitivity equations (as well as the differential equation (1),(2)) on the partition ρ^μ using a variety of differential equation solvers. Further, suppose that $[w_1^{\rho^\mu}(p_1, \theta), \dots, w_M^{\rho^\mu}(p_M, \theta)]$ is the approximation of $[y(p_1, \theta), \dots, y(p_M, \theta)]$. Define $w^{\rho^\mu} : [0, T] \times \Theta \rightarrow R^m$ to be the linear interpolation of the points $\{(p_j, w_j^{\rho^\mu}(p_j, \theta))\}$. Now suppose that $[s_1^{\rho^\mu}(p_1, \theta), \dots, s_M^{\rho^\mu}(p_M, \theta)]$ is the approximation of $[s_\theta(p_1, \theta), \dots, s_\theta(p_M, \theta)]$. Define $s^{\rho^\mu} : [0, T] \times \Theta \rightarrow R^{m \times p}$ to be the linear interpolation of the points $\{(p_j, s_j^{\rho^\mu}(p_j, \theta))\}$.

II. BOUNDED ERROR PARAMETER IDENTIFICATION

Bounded error parameter identification has been studied thoroughly for application to a variety of problems. Walter and Jaulin have done a considerable amount of work in the area and developed the SIVIA method (Set Inversion via Interval Analysis) which is a guaranteed method for bounded error parameter identification on non-linear

models [6]. The method maps small sets in the parameter space to the model output using interval analysis. If the set of parameters maps into the predetermined bounded regions in the model output then it is deemed feasible. The method produces inner sets and outer sets that bound the true set of parameters from above and below. Recently, Kieffer and Walter [8] considered a more restrictive guaranteed method for cooperative systems where the model is exact. These methods are extremely robust but they typically are also very time consuming and hence restrictive in application. Additionally, the complexity of the numerical problem increases greatly if more accuracy is desired.

The methods based in interval analysis were extended to models described by differential equations as seen in [5], [7], [9], and [10].

The method that we introduce in this paper also has restrictions because of additional assumptions on the dynamics, but when it can be applied, it is computationally efficient and accurate. Our method is different from the fore mentioned methods because uses the implicit function theorem to find the boundary of the set of admissible parameters. One of the numerical advantages of this method is that it works by solving differential equations at points in the parameter space instead of on intervals in the parameter space. When applicable, this method has the added benefit of providing insight to the problem of designing experiments.

In order to be precise, an experiment is defined to be a set of N sampling times with corresponding intervals and will be denoted

$$\begin{aligned} E^{N,m} &= \{E_{j,i}^{N,m}\}_{j=1 \dots N}^{i=1 \dots m} \\ &= \{(t_j, [a_j^i, b_j^i])\}_{j=1 \dots N}^{i=1 \dots m} \end{aligned}$$

Given an experiment $E^{N,m}$, we seek to find the set \mathcal{S} of admissible parameters that satisfy

$$\mathcal{S}(E^{N,m}, \Theta) = \{\theta \in \Theta \mid y_i(t_j, \theta) \in [a_j^i, b_j^i]\}$$

The following results provide the theoretical justification for the proposed bounded error parameter identification method. Theorem II.1 and Theorem II.2 are stated to provide the reader with insight on the derivation of the main result in Theorem II.3. The theorems given increase simultaneously in number and complexity. Theorem II.2 is a generalization of Theorem II.1, as Theorem II.3 is a generalization of Theorem II.2.

Theorem II.1. Let $E^{N,1} = \{(t_j, [a_j, b_j])\}_{j=1}^N$ be an experiment.

(i) Suppose that $y(t_j, \cdot) : \Theta \rightarrow [a_j, b_j]$ is a surjection and without loss of generality $\partial y(t_j, \theta)/\partial \theta_1 \neq 0$ for $\theta \in \Theta$ for $j = 1, \dots, N$.

Then there exists a compact set $Q \subset \mathbb{R}^{p-1}$, a compact set $V \subset Q$, and functions $l_{E^{N,1}} : V \rightarrow \mathbb{R}^1$ and $u_{E^{N,1}} : V \rightarrow \mathbb{R}^1$ such that

$$\mathcal{S}(E^{N,1}, K) = \{[\theta_1, q^T]^T | q \in Q \text{ and } \theta_1 \in [l_{E^{N,1}}(q), u_{E^{N,1}}(q)]\}$$

where

$$K = \{\theta \in \Theta | [\theta_2, \dots, \theta_p] \in Q\}$$

This theorem, implies that there are functions that describe the boundary of the set of admissible parameters for exact models. We will provide a sketch of the proof, the full proof can be found in the thesis [4].

Proof: For this method we are decomposing the parameter $\theta \in \mathbb{R}^p$ into two parts: $q_1 \in \mathbb{R}$ and $q_2 \in \mathbb{R}^{p-1}$, where $\theta = [q_1, q_2^T]^T$. We consider the function $Z : \mathbb{R}^1 \times \mathbb{R}^{p-1} \times [0, T] \times \mathbb{R}^1 \rightarrow \mathbb{R}^1$ defined by

$$Z(q_1, q_2, t, p) = y(t, [q_1, q_2^T]^T) - p. \quad (5)$$

Without loss of generality we suppose that $\partial y(t_j, \theta)/\partial \theta_1 > 0$ for $\theta \in \Theta$ and $j = 1, \dots, N$. By assumption (i), there exists a $\tilde{\theta}_j = [\tilde{q}_{1,j}, \tilde{q}_{2,j}^T]^T \in \Theta$ such that $Z(\tilde{q}_{1,j}, \tilde{q}_{2,j}, t_j, a_j) = 0$ and a $\hat{\theta}_j = [\hat{q}_{1,j}, \hat{q}_{2,j}^T]^T \in \Theta$ such that $Z(\hat{q}_{1,j}, \hat{q}_{2,j}, t_j, b_j) = 0$ for each $j = 1, \dots, N$. Then by the implicit function theorem there exists compact sets $Q_{a_j} \subset \mathbb{R}^{p-1}$ and $Q_{b_j} \subset \mathbb{R}^{p-1}$ and functions $z_{a_j} : Q_{a_j} \rightarrow \mathbb{R}$ and $z_{b_j} : Q_{b_j} \rightarrow \mathbb{R}$ such that

$$Z(z_{a_j}(q_2), q_2, t_j, a_j) = 0, \quad (6)$$

for all $q_2 \in Q_{a_j}$ and

$$Z(z_{b_j}(q_2), q_2, t_j, b_j) = 0, \quad (7)$$

for all $q_2 \in Q_{b_j}$. For each j and $q_2 \in Q_{a_j} \cap Q_{b_j}$, if $y(t, [q_1, q_2^T]^T) \in [a_j, b_j]$ then $q_1 \in [z_{a_j}(q_2), z_{b_j}(q_2)]$ since by assumption (i), $\partial y(t_j, \theta)/\partial \theta_1 > 0$. Let $Q = (\bigcap Q_{a_j}) \cap (\bigcap Q_{b_j})$ and define the set

$$V = \{q \in Q | \max_j z_{a_j}(q) \leq \min_j z_{b_j}(q)\}$$

Now define the function $l_{E^{N,1}} : V \rightarrow \mathbb{R}^1$ by

$$l_{E^{N,1}}(q) = \max_j z_{a_j}(q), \quad (8)$$

and the function $u_{E^{N,1}} : V \rightarrow \mathbb{R}^1$

$$u_{E^{N,1}}(q) = \min_j z_{b_j}(q). \quad (9)$$

Then the functions $l_{E^{N,1}}$ and $u_{E^{N,1}}$ define the boundary for the membership set. ■

For the majority of the realistic problems it is not possible to exactly compute these boundaries and some type of numerical approximation must be used. As noted in [4], it is possible to approximate the set of admissible parameters for exact models.

Theorem II.2. Let $E^{N,1} = \{(t_j, [a_j, b_j])\}_{j=1}^N$ be an experiment.

(i) Suppose that $y(t_j, \cdot) : \Theta \rightarrow [a_j, b_j]$ is a surjection and without loss of generality $\partial y(t_j, \theta)/\partial \theta_1 \neq 0$ for $\theta \in \Theta$ and $j = 1, \dots, N$.

Then there exists a set $S^{\pi^\delta, k}(E^{N,1}, K)$ such that

$$\lim_{\delta \rightarrow 0} \lim_{k \rightarrow \infty} S^{\pi^\delta, k}(E^{N,1}, K) = S(E^{N,1}, K)$$

Furthermore, $S^{\pi^\delta, k}(E^{N,1}, K)$ can be solved for explicitly.

In above theorem, π^δ is a δ -fine partition of Q and k is the number of iterations in Newton's method. To solve for the boundary functions, we approximate the functions z_{a_j} and z_{b_j} , defined by Equations (6) and (7), at each $r \in \pi^\delta$. To approximate z_{a_j} , we fix r , t_j , and a_j , and apply Newton's method to $Z(q_1, r, t_j, a_j)$ defined by Equation (5), with initial condition r_0 . This approximation will be denote by $N_k^{\pi^\delta}(r_0, r, t_j, a_j)$ and the interpolation of the points $\{(r, N_k^{\pi^\delta}(r_0, r, t_j, a_j))\}$ will be an approximation of the function z_{a_j} . From this construction we can define approximations of $u_{E^{N,1}}$ and $l_{E^{N,1}}$, from which we define $S^{\pi^\delta, k}(E^{N,1}, K)$. As $\delta \rightarrow 0$ and $k \rightarrow \infty$ the approximate boundary function approach the true boundary function of the set of admissible parameters.

The final theorem shows that we can numerically approximate the set of admissible parameters for approximate models and extends the result to vector models [4]. Here ρ^μ is a μ -fine partition of $[0, T]$.

Theorem II.3. Let $E^{N,m} = \{(t_j, [a_j^i, b_j^i])\}$ be an experiment and let $\{\rho^\mu\}$ be a set of partitions of $[0, T]$.

(ii) Suppose that for each ρ^μ , $j = 1, \dots, N$, and $i = 1, \dots, m$, $w_i^{\rho^\mu}(t_j, \cdot) : \Theta \rightarrow [a_j^i, b_j^i]$ is a surjection and without loss of generality is differentiable with respect to θ_1 . Further suppose that $\partial w_i^{\rho^\mu}(t_j, \theta) / \partial \theta_1 \neq 0$ for $\theta \in \Theta$, $j = 1, \dots, N$ and $i = 1, \dots, m$.

Then there exists a set $S^{\rho^\mu, \pi^\delta, k}(E^{N, m}, K)$ such that

$$\lim_{\delta \rightarrow 0} \lim_{k \rightarrow \infty} \lim_{\mu \rightarrow 0} S^{\rho^\mu, \pi^\delta, k}(E^{N, m}, K) = S(E^{N, m}, K).$$

Furthermore, $S^{\rho^\mu, \pi^\delta, k}(E^{N, m}, K)$ can be solved for explicitly.

To prove the final theorem we replace the function Z defined by Equation (10) with the function

$$Z_i^{\rho^\mu}(q_1, q_2, t, p) = w_i^{\rho^\mu}(t, [q_1, q_2^T]^T) - p. \quad (10)$$

We then proceed as we did in Theorems II.1 and II.2.

Finally we note that, although this method was developed for bounded error parameter estimation, it also provides a useful tool to deal with problems in the design of experiments. In particular, the method allows us to determine exactly which parameters are no longer admissible as more data is collected. This characteristic is useful in finding V-Optimal designs [4].

III. EXAMPLE 1: THE LOGISTIC EQUATION

We demonstrate the results of Theorem II.1 and Theorem II.3 applied to a model defined by the logistic equation. We chose this simple example because we can solve for the membership set analytically and compare it to the approximation defined by Theorem II.3. Additionally, we can generate nice graphical representations of the membership set for two parameter parameter identification problems. We consider a model described by logistic equation,

$$\dot{x}(t, \theta) = cx(t, \theta) - dx(t, \theta)^2, \quad (11)$$

$$x(0, \theta) = 0.1, \quad (12)$$

with observation given by

$$y(t, \theta) = x(t, \theta). \quad (13)$$

Here, $\theta \in \Theta = \{[d, c]^T \in \mathbb{R}_+^2 | c/d > .1\}$, $t \in [0, 20]$, and $x : [0, 20] \times \Theta \rightarrow \mathbb{R}_+^1$.

For all $\theta \in \Theta$ and $t \in [0, 20]$, $\partial y(t, \theta) / \partial d < 0$. Furthermore, the model is onto the interval in the models output given in the experiment

$$E^{1,1} = \{(3, [y(3, [.5, 1]^T) - .1, y(3, [.5, 1]^T) + .1])\}.$$

For this experiment we can explicitly solve for the function

$$z_{a_1}(c) = \frac{ce^{ct_1 - a_1/x_0}}{a_1(-1 + e^{ct_1})}, \quad (14)$$

where $c \in [.85, 1.15]$. Similarly we can explicitly solve for the function

$$z_{b_1}(c) = \frac{ce^{ct_1 - b_1/x_0}}{b_1(-1 + e^{ct_1})}, \quad (15)$$

This is a very special example and in most practical problems we cannot solve the differential equation exactly so we can not solve for the membership set analytically. We will now proceed by approximating the membership set using only the differential equation and then compare the approximation to the analytical result we obtained. To approximate the membership set we will first approximate the model described in Equations (11) and (12) using the Runge Kutta method on the partition $\rho^{0.1} = [0, 0.1, 0.2, \dots, 19.9, 20]$. Using this approximation we can find a corresponding approximation for z_{a_1} and z_{b_1} at the points of a partition of Q defined by $\pi^{0.1} = [0.85, 0.95, 1.05, 1.15]$. Finding to the roots to Equation (10) on this partition with respect to the experiment $E^{1,1}$ using 8 iterations of Newton's method leads to the approximation $S^{\rho^{0.1}, \pi^{0.1}, 8}(E^{1,1}, K)$ seen in Fig. 1.

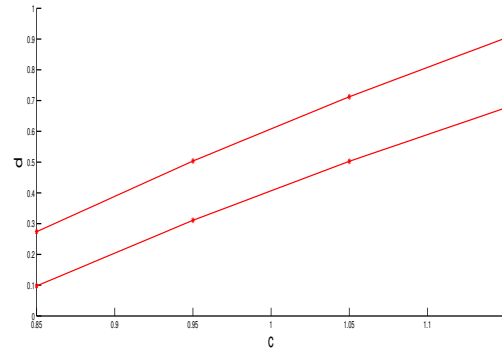


Fig. 1. $S^{\rho^{0.1}, \pi^{0.1}, 8}(E^{1,1}, K)$

With a modest computational cost we are able to find an approximation $S^{\rho^{0.1}, \pi^{0.1}, 8}(E^{1,1}, K)$ that satisfies

$$d_H \left(S^{\rho^{0.1}, \pi^{0.1}, 8}(E^{1,1}, K), S(E^{1,1}, K) \right) < 10^{-3}$$

where $d_H(\cdot, \cdot)$ is the Hausdorff metric.

Next we consider the experiment

$$\begin{aligned}
E^{2,1} &= \{E_1^{2,1}, E_2^{2,1}\} \\
&= \{(3, [y(3, [.5, 1]^T) - .1, y(3, [.5, 1]^T) + .1]), \\
&\quad (15, [y(15, [.5, 1]^T) - .1, y(15, [.5, 1]^T) + .1])\}.
\end{aligned}$$

Fig. 2 shows a graphical representation of a numerical approximation of $\mathcal{S}(E_1^{2,1}, K)$, the area between the red curves, and $\mathcal{S}(E_2^{2,1}, K)$, the area between the blue curves. The intersection of these two sets is an approximation of the set $\mathcal{S}(E^{2,1}, K)$. Additionally it shows the function $l_{E^{2,1}}$, the black dashed curve, and $u_{E^{2,1}}$, the black solid curve, that define the boundary of the membership set.

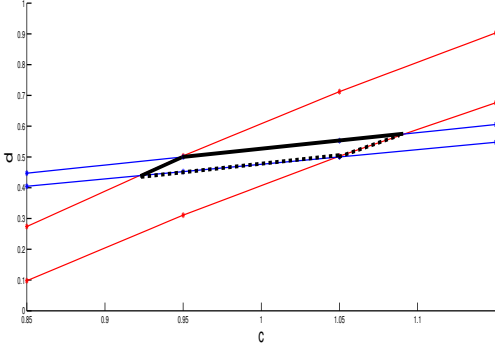


Fig. 2. An approximation of $\mathcal{S}(E^{2,1}, K)$ showing $l_{E^{2,1}}$ (black dashed) and $u_{E^{2,1}}$ (black solid)

This example is admittedly simple but this method can be implemented on problems where the model has vector output and there are more parameters to be identified. As with any bounded error parameter identification method, the time required to solve for the membership set increases exponentially with the number of unknown parameters.

IV. BOUNDED ERROR PARAMETER IDENTIFICATION FOR DELAY DIFFERENTIAL EQUATIONS

We are working to extend this method (both theory and computation) to distributed parameter models such as those described by partial and delay differential equations. Validating parameters for such models is considerably more difficult than models based on ordinary differential equations (ODEs) because of issues of identifiability. For example, the asymptotic theory that corresponds to least squares parameter estimation and maximum likelihood parameter estimation can not be used for on most DDE models because they do not satisfy the hypothesis prerequisite to the theory ([3], [11]). However, the following example illustrates that this method might

be useful in the distributed parameter setting.

Consider Hutchison's equation

$$\dot{x}(t, \theta) = ax(t, \theta)\left(1 - \frac{1}{2}x(t-r, \theta)\right), \quad 0 < t < T,$$

with initial data

$$x(0, \theta) = 1 \quad \text{and} \quad x(s, \theta) = 1, \quad -r < s < 0$$

and $\theta = [a, r]^T \in \Theta = [0, 2] \times [.9, 1.1]$. Given a set of data, $\{(t_j, d_j)\}_{j=1}^N$ we have two basic goals. The first is to identify the unknown parameter θ and the second goal is to validate the estimate. Given the data set $\{(3.589, 2.048), (6.136, 2.004)\}$, we use ordinary least squares to estimate the unknown parameter. For different initial guesses the gradient based Levenberg-Marquardt optimization method produced the values $\theta^1 = [1, 1]^T$, $\theta^2 = [.6, 1]^T$ and $\theta^3 = [0.9454, 0.9747]^T$ as minimizers of the ordinary least squares cost function. Fig. 3 shows the models that fit the data set.

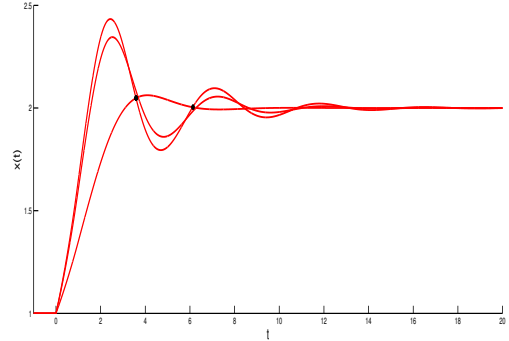


Fig. 3. The data and the possible models.

Suppose we further know that the data point is within 1% of the true value. From this information we can define the experiment

$$\begin{aligned}
E^{2,1} &= \{E_1^{2,1}, E_2^{2,1}\} \\
&= \{(3.589, [2.068, 2.088]), (6.136, [1.984, 2.024])\}
\end{aligned}$$

and using bounded error parameter estimation we can construct the set in the parameter space that satisfies the experiment. For $Q = [.9, 1.1]$, $\mathcal{S}(E_1^{2,1}, K)$ is approximated by the area between the blue curves above $a = .8$ and the area between the blue curves below $a = .8$ in Fig 8. Also, $\mathcal{S}(E_2^{2,1}, K)$ is approximated by the area between the red curves above $a = .8$ and the area between the red curves

below $a = .8$ in Fig. 4. The approximation of $\mathcal{S}(E^{2,1}, K)$ is the intersection of $\mathcal{S}(E_1^{2,1}, K)$ and $\mathcal{S}(E_2^{2,1}, K)$.

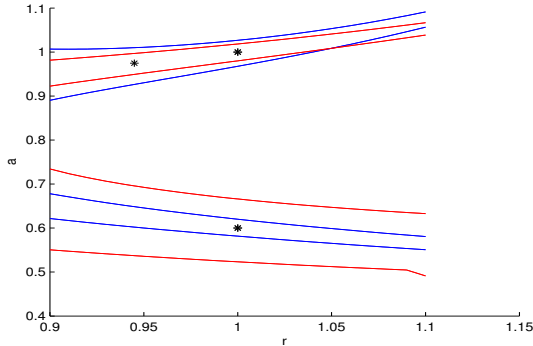


Fig. 4. The numerical approximation of $\mathcal{S}(E_1^{2,1}, K)$ and $\mathcal{S}(E_2^{2,1}, K)$

The next question is, given one more sample can we identify the parameter with some confidence? For certain times the answer is no. For example, if we sample at equilibrium we gain no new information. However, if we sample at $t = 2$ we can eliminate all but one of the estimates and use bounded error parameter identification to validate the result. Therefore, we define a new experiment to be

$$\begin{aligned} E^{3,1} &= \{E_1^{3,1}, E_2^{3,1}, E_3^{3,1}\} \\ &= \{(3.589, [2.068, 2.088]), (6.136, [1.984, 2.024]), \\ &\quad (2, [1.605, 1.635])\} \end{aligned}$$

This new information eliminates all but one of the parameter estimates and $\mathcal{S}(E^{3,1}, K)$ is considerably smaller than $\mathcal{S}(E^{2,1}, K)$. The area between the green curves that intersects the area between the blue curves and the red curves in Fig. 5 show estimates of $\mathcal{S}(E^{3,1}, K)$.

There is still considerable work to be done in the area of bounded error parameter estimation for delay differential equations but the numerical results presented here illustrate that the method is potentially useful for parameter identification and validation for such systems.

V. CONCLUSION

Bounded error parameter identification can be a valuable tool for validating parameter estimates from small data sets. In general, given a set of bounded error data, constructing the set of admissible parameters is very costly. Guaranteed methods for bounded error parameter estimation are robust but time consuming. However, imposing additional requirement on the model can improve the speed and accuracy of the method for models that satisfy these additional

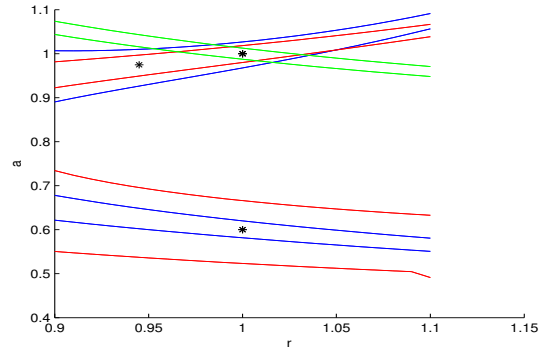


Fig. 5. An estimate of $\mathcal{S}(E^{3,1}, K)$

constraints. Future efforts will focus on applying the idea to more general models.

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