Upwind Approximations and Mesh Independence for LQR Control of Convection Diffusion Equations

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***Abstract*—The development of practical computational schemes for optimization and control of non-normal distributed parameter systems requires that one builds certain computational efficiencies (such as mesh independence) into the approximation scheme. We consider some numerical issues concerning the application of Kleinman-Newton algorithms to discretizations of infinite dimensional Riccati equations that arise in control of PDE systems. We show that dual convergence and compactness play central roles in both convergence and mesh independence and we present numerical results to illustrate the theory.**

I. INTRODUCTION

In this paper we consider some numerical issues concerning the application of Kleinman-Newton algorithms to discretizations of infinite dimensional Riccati equations. In particular, we focus on mesh independence for standard finite element and stabilizing Petrov-Galerkin approximations of convection diffusion equations. Since these problems are not self-adjoint one must deal with dual convergence (see [3], [4] and [10]). In addition, we discuss the case where the "Q" operator, see (11) and (13), is not Hilbert-Schmidt and provide examples to illustrate some potential problems. We employ simple first order upwinding in the computation of functional gains for LQR feedback control when the convection term dominates. Upwinding is a stabilization scheme that is typically used to avoid oscillatory solutions in the simulation of such equations. However, it has also been observed that numerical oscillations occur when standard finite element or finite difference schemes are applied to the Riccati equations that define the feedback operators (see [12] and [15]). Although upwinding can eliminate (or reduce) the numerical oscillations, it can also produce inaccurate solutions. The upwind method considered here can be formulated as a Petrov-Galerkin approximation and this formulation can help in the selection of an "optimal" stabilization parameter. Krueger [12] used high order stabilized finite element schemes to address this issue. Here we focus on the simple first order scheme (see [7]) and investigate the connections between the choice of stabilization parameter and mesh independence of the Kleinman-Newton algorithm (see [1], [2] and [4]). We review the standard Galerkin piecewise linear finite element approximation and introduce the upwind approximation. We discuss convergence of the schemes, outline the mesh independence principle and present numerical results to illustrate the ideas.

II. PROBLEM SETTING AND BASICS

We consider the controlled convection diffusion equation

$$\frac{\partial w(t,x)}{\partial t} = \mu \frac{\partial^2 w(t,x)}{\partial x^2} + \nu \frac{\partial w(t,x)}{\partial x} + b(x)u(t), \qquad (1)$$

for $t > 0, \quad 0 < x < 1$ with boundary conditions

$$w(t,0) = 0, \quad w(t,1) = 0, \qquad (2)$$

and initial condition

$$w(0,x) = w_0(x). \qquad (3)$$

The *control input* is the function $u(t)$. Here, $\mu > 0$ and $b(\cdot) \in H = L^2(0,1)$ is a given function. When the Peclet number $Pe = \frac{\nu}{\mu}$ is large, the problem is convection dominated and requires special numerical techniques. Unless otherwise stated we shall set $\nu = 1$.

Define the convection diffusion operator, A_μ on $H = L^2(0,1)$ with domain

$$\mathcal{D}(A_\mu) = \left\{ \varphi(\cdot) \in H^2(0,1) : \varphi(0) = 0, \quad \mu\varphi(1) = 0 \right\} \qquad (4)$$

by

$$\left[A_\mu \varphi(\cdot)\right](x) = \mu \frac{d^2\varphi(x)}{dx^2} + \frac{d\varphi(x)}{dx}, \quad \forall \varphi(\cdot) \in \mathcal{D}(A_\mu). \qquad (5)$$

The Hilbert adjoint of A_μ, under the standard $L^2(0,1)$ inner product, is given by

$$\left[A_\mu^* \varphi(\cdot)\right](x) = \mu \frac{d^2\varphi(x)}{dx^2} - \frac{d\varphi(x)}{dx}, \quad \forall \varphi(\cdot) \in \mathcal{D}(A_\epsilon^*) \qquad (6)$$

where

$$\mathcal{D}(A_\mu^*) = \mathcal{D}(A_\mu) = \left\{ \varphi(\cdot) \in H^2(0,1) : \mu\varphi(0) = 0 = \varphi(1) \right\}. \qquad (7)$$

We note that for $\mu > 0$, $\mathcal{D}(A_\mu^*) = H_0^1(0,1) \cap H^2(0,1)$ but as $\mu \to 0$ the problem becomes highly non-normal and computation of the functional gains becomes more difficult.

In order to complete the formulation of the control system, we define the linear operator $B : R^1 \to H = L^2(0,1)$ by

$$\left[Bu\right](x) = b(x)u$$

and note that B is a compact linear operator with finite rank. If one defines the *state space* $H = L^2(0,1)$, then the controlled convection diffusion equation (1)-(3) is equivalent to the system

$$\dot{z}(t) = A_\mu z(t) + Bu(t), \qquad z(0) = z_0 \in L^2(0,1) \qquad (8)$$

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in $L^2(0,1)$. Let $C : H = L^2(0,1) \to Y$ be a bounded linear operator into the Hilbert space Y and consider the controlled output

$$y(t) = [Cz(t)]\,. \tag{9}$$

We do not assume that C is compact.

Define the quadratic cost function $J(u)$ by

$$J(u) = \int_0^\infty \left(\|y(s)\|^2 + \|u(s)\|^2\right)\, ds, \tag{10}$$

where $z(s)$ is the solution to (8) for a given control $u \in L^2(0,\infty;U)$. The LQR control problem is to minimize the quadratic cost $J(u)$ over all controls $u \in L^2(0,\infty;U)$. If $Q : L^2(0,1) \to L^2(0,1)$ is defined by $Q = C^*C$, then the cost function becomes

$$J(u) = \int_0^\infty \left(\langle Qz(s), z(s)\rangle + \|u(s)\|^2\right)\, ds \tag{11}$$

and it is well known (see [3], [5] and [10]) that the optimal control is given by state feedback

$$u_{opt} = -Kz(t) = -\int_0^1 k(s)w(t,s)ds \tag{12}$$

where the kernel $k(s)$ is called the functional gain. The optimal feedback gain operator $K : L^2(0,1) \to R$ is given by

$$K = B^*\Pi$$

where $0 \le \Pi = \Pi^* \in \mathcal{L}(H)$ is the solution of the abstract algebraic Riccati operator equation

$$\mathcal{F}(\Pi) \triangleq A_\mu^*\Pi + \Pi A_\mu - \Pi BB^*\Pi + Q = 0. \tag{13}$$

Consider a sequence of approximating problems defined by (H^N, A_μ^N, B^N, C^N), where $H^N \subset H$ is a sequence of finite dimensional subspaces of H and $A_\mu^N \in \mathcal{L}(H^N, H^N)$, $B^N \in \mathcal{L}(U, H^N)$ and $C^N \in \mathcal{L}(H^N, Y)$ are bounded linear operators. Let $P^N : H \to H^N$ denote the orthogonal projection of H onto H^N satisfying $\left\|P^N\right\| \le 1$. Also, we assume that for all $x \in H$, $\|P^N x - Px\| \to 0$ as $N \to \infty$. An approximating sequence produces a finite dimensional approximate Riccati equation of the form

$$\begin{aligned}\mathcal{F}^N(\Pi) \quad &\triangleq \quad [A_\mu^N]^*\Pi^N + \Pi^N[A_\mu^N] - \Pi^N[B^N][B^N]^*\Pi^N \\ &\qquad +Q^N = 0. \end{aligned}\tag{14}$$

There are two basic issues to be considered when constructing approximations to the Riccati equation (13).

1) *Convergence under mesh refinement.* In order for the approximation scheme to be useful, one needs to establish that $\Pi^N \to \Pi$ as $N \to \infty$ in some sense. This problem has been the subject of numerous papers for the past twenty years. Typically, one can obtain strong convergence with mild assumptions on the approximation scheme. Even this type of result requires "dual convergence" of the approximating "A" operators. In particular, one needs to establish strong convergence of the semi-groups $S_\mu^N(t) \triangleq e^{A_\mu^N t}$ to $S(t) \triangleq e^{A_\mu t}$ and the duals $[S_\mu^N(t)]^*$ to $S^*(t)$ (see [3], [4] and [10]). However, additional requirements such as compactness of the operators B and C are needed in order to obtain uniform operator convergence $\left(\left\|\Pi^N - P^N\Pi P^N\right\| \to 0\right)$. The papers [4], [10], [13] and [14] provide a nice summary of such results that apply to convection diffusion equations. For the problem above, B is compact so the only issue is the compactness of the output operator C. We discuss this point later.
2) *Mesh independence and numerical conditioning of the approximating Riccati equation.* Strictly speaking, mesh independence (see [1], [2] and [4]) does not make sense unless the approximating scheme produces uniform operator convergence. However, sometimes it is possible to overcome this technical issue by establishing norm convergence of the gain operators and discussing mesh independence within this context. Although we briefly discuss this issue below, a more complete analysis of this problem is not possible in this short paper. Finally, if the problem is convection dominated, then additional numerical issues arise.

We use the upwind Petrov-Galerkin method in [7] to approximate the Riccati equation (13) and produce approximations of the functional gain $k(s)$. A Kleinman-Newton algorithm is used to solve the finite dimensional Riccati equation. Mesh independence and numerical conditioning of these algorithms are studied as a function of the stabilization parameter.

III. THE STANDARD FINITE ELEMENT - GALERKIN SCHEME

The basic numerical scheme is based on the standard linear finite element Galerkin approximation of the system (1)-(3). The basic idea is to multiply both sides of the convection diffusion equation (1) by a test function $\psi(\cdot) \in H_0^1(0,1)$, integrate by parts and apply the boundary conditions (2) to yield the equation

$$\begin{aligned}\int_0^1 \frac{\partial w(t,x)}{\partial t}\psi(x)dx \quad &= \quad -\mu\int_0^1 \frac{\partial w(t,x)}{\partial x}\psi'(x)dx \\ &\qquad + \int_0^1 \frac{\partial w(t,x)}{\partial x}\psi(x)dx \\ &\qquad + \left[\int_0^1 b(x)\psi(x)dx\right]u(t).\end{aligned}\tag{15}$$

Thus, for all $\psi(\cdot) \in H_0^1(0,1)$ the variational equation (15) must hold and we say that $w(t,x)$ is a *weak solution* of (1)-(2) if (15) holds for all $\psi(\cdot) \in H_0^1(0,1)$ and $w(0,x) = w_0(x)$ almost everywhere on $(0,1)$.

If one defines the bilinear form $a_\mu(\cdot,\cdot)$ on $H_0^1(0,1) \times H_0^1(0,1)$ by

$$a_\mu(w(\cdot),\psi(\cdot)) = \int_0^1 \left[\mu\frac{dw(x)}{dx}\frac{d\psi(x)}{dx} + \frac{dw(x)}{dx}\psi(x)\right]dx,$$

for all $w(\cdot)$, $\psi(\cdot)$ in $H_0^1(0,1)$, then the variational equation

(15) can be written as

$$\left\langle \frac{\partial w(t,\cdot)}{\partial t}, \psi(\cdot) \right\rangle_{L^2(0,1)} = -a_\mu(w(\cdot),\psi(\cdot)) + \left[\langle b(\cdot), \psi(x\cdot)\rangle_{L^2(0,1)} \right] u(t)$$

for all $\psi(\cdot) \in H_0^1(0,1)$.

The next step in approximating A_μ is to choose a finite dimensional subspace of $H_0^1(0,1)$ and project onto this space. Let $x_i = \frac{i}{N+1}$, $i = 0,1,\ldots,N+1$ so that $\Delta x = \frac{1}{N+1}$ and define the piecewise linear continuous splines $h_i^N(\cdot)$, $i = 1,\ldots,N$, on the interval $[0,1]$ by

$$h_i^N(x) = \begin{cases} \frac{1}{N}(x - x_{i-1}^N), & x \in [x_{i-1}^N, x_i^N], \\ \frac{1}{N}(x_{i+1}^N - x), & x \in [x_i^N, x_{i+1}^N], \\ 0 & \text{otherwise.} \end{cases} \tag{16}$$

The *finite element space* is the space $V_1^N \subset H_0^1(0,1)$ defined by

$$V_1^N \equiv span\{h_i^N(\cdot)\}_{i=1}^N.$$

If $a_\mu^N(w(\cdot),v(\cdot))$ is the bilinear form $a_\mu(w(\cdot),v(\cdot))$ restricted to the finite element space V_1^N, then $a_\mu^N(w(\cdot),v(\cdot))$ defines a bounded linear operator F_μ^N on V_1^N by

$$\left\langle \left[F_\mu^N w(\cdot)\right], \psi(\cdot)\right\rangle_{L^2(0,1)} = a_\mu^N(w(\cdot),\psi(\cdot)).$$

Let $P^N : L^2(0,1) \to V_1^N$ be the orthogonal projection onto V_1^N and $b^N(\cdot)$ and $w_0^N(\cdot)$ the orthogonal projections of $b(\cdot)$ and $w_0(\cdot)$ onto V_1^N, respectively. In particular, $b^N(\cdot) = P^N b(\cdot)$, $w_0^N(\cdot) = P^N w_0(\cdot)$ and we define $G^N : R^1 \to V_1^N$ by

$$\left[G^N u\right](x) = b^N(x)u.$$

The finite element Galerkin approximation to the system (8) is the finite dimensional system defined in V_1^N by

$$\begin{aligned} P^N\left[\frac{\partial}{\partial t} w(t,x)\right] &= F_\mu^N w(t,x) + G^N u(t) \\ P^N w(0,x) &= P^N w_0(x) \in V_1^N. \end{aligned} \tag{17}$$

If $P^N w(t,x) = w^N(t,x) = \sum_{i=1}^N z_i(t) h_i^N(x)$, and one uses the test functions $\psi_i^N(x) = h_i^N(x)$, then the system that defines the evolution of the coefficients $z^N(t) = [z_1(t), z_2(t), \ldots, z_N(t)]^T$ is given by the matrix equation

$$\mathbf{M}^N \dot{z}^N(t) = \mathbf{F}_\mu^N z^N(t) + \mathbf{G}^N u(t),$$

where $\mathbf{M}^N$ is the mass matrix, $\mathbf{F}_\mu^N$ is the stiffness matrix, $\mathbf{G}^N$ is the control input matrix defined by

$$\mathbf{G}^N = \left[\begin{array}{cccc} \mathbf{G}_1^N & \mathbf{G}_2^N & \cdots & \mathbf{G}_N^N \end{array}\right]^T$$

and for $i = 1,2,\ldots,N$, $\mathbf{G}_i^N = \langle b(\cdot), h_i^N(\cdot)\rangle_{L^2(0,1)}$.

Clearly, the matrix representation of the approximate operators A_μ^N and B^N are given by

$$\mathbf{A}_\mu^N = \left[\mathbf{M}^N\right]^{-1}\mathbf{F}_\mu^N \quad \text{and} \quad \mathbf{B}^N = \left[\mathbf{M}^N\right]^{-1}\mathbf{G}^N,$$

respectively. Therefore, the finite element Galerkin approximation of the control system for the convection diffusion equation is given by the finite dimensional system

$$\dot{z}^N(t) = \mathbf{A}_\mu^N z^N(t) + \mathbf{B}^N u(t). \tag{18}$$

The classical *simulation* problem is defined when the control function $u(t)$ and the initial data $w_0(x)$ are **given** and the finite element model (18) is integrated to generate the coefficients $z^N(t) = [z_1(t), z_2(t), \ldots, z_N(t)]^T$. This problem has a long history and is well understood. However, if the finite element matrices $\mathbf{A}_\mu^N$ and $\mathbf{B}^N$ are used to design a (feedback) controller, then additional convergence issues arise and require separate analysis depending on the particular design problem. For example, if one is attempting a LQR design then Banks and Kunisch [3] have shown that the system $\left[\mathbf{A}_\mu^N, \mathbf{B}^N\right]$ must be stabilizable (uniformly in N). Other conditions are needed to guarantee the mesh independence principle holds (see [4]).

IV. UPWIND SCHEME

In the simplest form, upwinding can be thought of as approximating the convection term by a backward difference operator. However, in the Petrov-Galerkin formulation one sees that upwinding can be viewed as introducing an additional diffusive term and controlling this term to improve accuracy (see [7]). In particular, we modify test functions by adding a quadratic element. As above, let $x_i = \frac{i}{N+1}$, $i = 0,1,\ldots,N+1$ so that $\Delta x = \frac{1}{N+1}$ and define the piecewise quadratic continuous splines $\gamma_i^N(\cdot)$, $i = 1,\ldots,N$, on the interval $[0,1]$ by

$$\gamma_i^N(x) = \begin{cases} -3(x - x_{i-1}^N)(x - x_i^N), & x \in [x_{i-1}^N, x_i^N], \\ +3(x - x_i^N)(x - x_{i+1}^N), & x \in [x_i^N, x_{i+1}^N], \\ 0 & \text{otherwise.} \end{cases} \tag{19}$$

For $\alpha \geq 0$, let

$$\psi_i^N(\cdot) = h_i^N(\cdot) + \alpha\gamma_i^N(\cdot) \tag{20}$$

be test functions. If $w^N(t,x) = \sum_{i=1}^N z_i(t) h_i^N(x)$, then the Petrov-Galerkin approximation of the convection diffusion equation comes from the system

$$\begin{aligned} \int_0^1 \frac{\partial w^N(t,x)}{\partial t}\psi_j^N(x)dx &= -\mu\int_0^1 \frac{\partial w^N(t,x)}{\partial x}[\psi_j^N(x)]' dx \\ &\quad + \int_0^1 \frac{\partial w^N(t,x)}{\partial x}\psi_j^N(x)dx \\ &\quad + \left[\int_0^1 b(x)\psi_j^N(x)dx\right]u(t), \end{aligned}$$

for $j = 1,2,\ldots,N$.

The mass and stiffness matrices for the upwind scheme are perturbations of the standard finite element mass and stiffness matrices and are constructed by adding terms that arise from the quadratic terms

$$\mathbf{M}_\alpha^N(w(\cdot),v(\cdot)) = \alpha\int_0^1 \left[h_i^N(x)\gamma_j^N(x)\right]dx$$

and

$$\mathbf{F}_{\alpha}^{N}(w(\cdot),v(\cdot))=\alpha\int_0^1\left[\frac{dh_i^N(x)}{dx}\frac{d\gamma_j^N(x)}{dx}\right]dx,$$

respectively. Here, α is a stabilization parameter that can be used to control the error and depends on the mesh used to generate the finite element system. Likewise, the control operator is approximated as above by adding the additional terms defined by the quadratic part of the test function. In particular, let

$$\tilde{\mathbf{G}}^N=\left[\begin{array}{cccc}\tilde{\mathbf{G}}_1^N & \tilde{\mathbf{G}}_2^N & \cdots & \tilde{\mathbf{G}}_N^N\end{array}\right]^T$$

where for $i=1,2,\ldots,N$, $\tilde{\mathbf{G}}_i^N=\left\langle b(\cdot),\gamma_i^N(\cdot)\right\rangle_{L^2(0,1)}$ and set

$$\mathbf{G}_{\alpha}^N=\mathbf{G}^N+\alpha\tilde{\mathbf{G}}^N.$$

It is instructive to look at the upwind system defined by the Petrov-Galerkin approximation above. For no control the system has the form

$$\begin{aligned}&\left[(\tfrac{1}{6}+\tfrac{\alpha}{4})\dot{w}_{k-1}(t)+(\tfrac{2}{3})\dot{w}_k(t)+(\tfrac{1}{6}-\tfrac{\alpha}{4})\dot{w}_{k+1}(t)\right]\\ &\quad=\frac{\mu}{\Delta x^2}\left[w_{k-1}(t)-2w_k(t)+w_{k+1}(t)\right]\\ &\qquad+\left[(w_{k+1}(t))-w_{k-1}(t))\right]/(2\Delta x)\\ &\qquad+\frac{\alpha v}{2\Delta x}\left[w_{k-1}(t)-2w_k(t)+w_{k+1}(t)\right]\end{aligned}$$

and one sees that the upwind approximation essentially introduces additional dissipativeness and modifies the mass matrix. Also, it is clear that as $\alpha\to 0$ the upwind method approaches the finite element system. The choice of the stabilization parameter α is crucial in obtaining good results. For the steady state problem with no control, the optimal parameter is known to be $\alpha_{opt}=\Delta x/6$. For the time dependent problem and the control problem above the choice of an optimal parameter is not well understood (see [7] and [12]).

V. CONVERGENCE AND MESH INDEPENDENCE THEOREM

There are two basic aspects of the Mesh Independence Principle (MIP) for Newton type methods (see [1] and [2]). Roughly speaking, the MIP may be broken down into convergence under mesh refinement and Newton iteration counts on a given mesh. Let $\mathcal{F}:D(\mathcal{F})\subseteq E\longrightarrow E$ be a nonlinear operator on an infinite dimensional Hilbert space E and consider the equation

$$\mathcal{F}(\Pi)=0. \tag{21}$$

Let $E^N\subseteq E$ be a sequence of finite dimensional approximating spaces and consider the sequence of discretized equations

$$\mathcal{F}^N(\Pi^N)=0, \tag{22}$$

where $\mathcal{F}^N:D(\mathcal{F}^N)\subseteq E^N\longrightarrow E^N$. Assume that (21) and (22) have unique solutions $\Pi_\infty\in D(\mathcal{F})$ and $\Pi_\infty^N\in D(\mathcal{F}^N)$, respectively. We say that the approximation scheme *converges* if

$$\lim_{N\to+\infty}\left\|\Pi_\infty^N-P^N\Pi_\infty P^N\right\|_{E^N}=0, \tag{23}$$

where $P^N:E\longrightarrow E^N$ is the projection of E onto E^N. Now assume that one applies a Newton type algorithm to (21) and (22) which produces (quadratically) convergent iterations Π_k and Π_k^N, $k=1,2,\ldots$.

For a given $\varepsilon>0$, $\Pi_0\in D(\mathcal{F})$ and $\Pi_0^N\in D(\mathcal{F}^N)$ define the numbers $M(\varepsilon,\Pi_0)$ and $M^N(\varepsilon,\Pi_0^N)$ by

$$M(\varepsilon,\Pi_0)\triangleq\inf\{k:\|\Pi_k-\Pi_\infty\|<\varepsilon\} \tag{24}$$

and

$$M^N(\varepsilon,\Pi_0^N)\triangleq\inf\{k:\left\|\Pi_k^N-\Pi_\infty^N\right\|<\varepsilon\}, \tag{25}$$

respectively. Here, Π_0 and Π_0^N are the starting values for the iterations. The (strong) MIP (see Theorem 2.1 in [2]) applied to the Riccati equation takes the form

$$M(\varepsilon,\Pi_0)=M^N(\varepsilon,P^N\Pi_0)+\tau(N), \tag{26}$$

where $\tau(N)\longrightarrow 0$ as $N\longrightarrow+\infty$.

In addition, assume there are minimal constants c and c^N (independent of k) such that

$$\|\Pi_{k+1}-\Pi_k\|\le c\,\|\Pi_k-\Pi_{k-1}\|^2$$

and

$$\left\|\Pi_{k+1}^N-\Pi_k^N\right\|\le c^N\left\|\Pi_k^N-\Pi_{k-1}^N\right\|^2,$$

respectively. One (strong) form of the MIP would be the condition that

$$c^N=c+\gamma(N), \tag{27}$$

where $\gamma(N)\longrightarrow 0$ as $N\longrightarrow+\infty$ (see [2]).

The following assumptions are essential to most of the results involving convergence and mesh independence for infinite dimensional systems.

Assumption 1 *For each $x\in H$, $S^N(t)P^Nx\longrightarrow S(t)x$ and the convergence is uniform in t on bounded subintervals of $[0,+\infty)$, for each $u\in U$, $B^Nu\longrightarrow Bu$ and for each $x\in H$, $C^NP^Nx\longrightarrow Cx$.*

Assumption 2 *For each $x\in H$, $[S^N(t)]^*P^Nx\longrightarrow S^*(t)x$ and the convergence is uniform in t on bounded subintervals of $[0,+\infty)$, for each $x\in H$, $[B^N]^*P^Nx\longrightarrow B^*x$ and for each $y\in Y$, $[C^N]^*y\longrightarrow C^*y$.*

Assumption 3 *The family of pairs (A^N,B^N) is uniformly stabilizable and the family of pairs (A^N,C^N) is uniformly detectable.*

Assumption 4 *The operators B and C are compact.*

The assumptions above (along with various technical conditions involving the smoothing property of the semigroup) can be used to establish norm convergence of the Riccati operators. There are a number of results along this line (see [6], [8], [9], [10], [13] and [14]). In addition, mesh independence follows from these same conditions (see [4]). The following result follows from the general results in [4] and involves establishing which conditions above hold for the finite element and upwind schemes.

Theorem 5 *The finite element scheme and upwind scheme with* $\alpha = \Delta x/6$ *satisfy Assumptions 1, 2 and 3. If in addition Assumption 4 holds, then there exist a constant* c *and* $\delta(N) \longrightarrow 0$ *as* $N \longrightarrow +\infty$ *such that*

$$\|\Pi_{k+1}^N - P^N\Pi_\infty P^N\| \leq (c+\delta(N))\|X_k^N - X_\infty^N\|^2 + \Delta^N, \tag{28}$$

where $\Delta^N \longrightarrow 0$.

Observe that (28) implies both convergence and mesh independence. Since the Newton method converges for the finite dimensional Riccati equation (14), one obtains uniform operator convergence of the Riccati operators. The triangle inequality implies mesh independence. It is important to note that the assumption on compactness, Assumption 4, can not be ignored as is illustrated in the following example.

Example: Let $H = R \times L^2$ and $A = \begin{bmatrix} -1 & 0 \\ 0 & -I \end{bmatrix}$ where I is the identity on L^2. Let $B = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$ and $C = \begin{bmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{2}I \end{bmatrix}$ so that $Q = C^*C = \begin{bmatrix} 3 & 0 \\ 0 & 2I \end{bmatrix}$. Since $B^*B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ it follows that $\Pi_\infty = \begin{bmatrix} 1 & 0 \\ 0 & I \end{bmatrix}$ is the solution to the Riccati equation

$$\mathcal{F}(\Pi) \triangleq A^*\Pi + \Pi A - \Pi BB^*\Pi + Q = 0.$$

Let $H^N = R \times R^N$ and define $P^N : H \to H^N$ to be the natural projection onto H^N. If $A^N = P^N A$, $B^N = P^N B$ and $C^N = CP^N$, then all the conditions in the previous theorem are satisfied except that C is not compact. The solution to the finite dimensional Riccati equation

$$\begin{aligned} \mathcal{F}^N(\Pi) \quad \triangleq \quad & [A^N]^*\Pi^N + \Pi^N[A^N] - \Pi^N[B^N][B^N]^*\Pi^N \\ & + Q^N = 0 \end{aligned}$$

is $\Pi_\infty^N = \begin{bmatrix} 1 & 0 \\ 0 & I^N \end{bmatrix}$ where I^N is the identity on R^N. Clearly, Π_∞^N does not converge to Π_∞ in the uniform operator norm since I is not compact. It is interesting to note that the feedback gain operators

$$K^N = [B^N]^*\Pi_\infty^N = \begin{bmatrix} 1 & 0 \end{bmatrix} = K \tag{29}$$

converge uniformly. This situation occurs in many problems and can be exploited to address mesh independence issues. We shall discuss this issue in a future paper. We now turn to some numerical examples.

VI. NUMERICAL EXAMPLES

We focus on two issues: (1) The impact of upwinding on convergence of the functional gains, mesh independence of the Kleinman-Newton algorithm and conditioning of the Riccati equations and (2) The role that the compactness of C plays on convergence and mesh independence. Note that if C is compact, then Q is Hilbert-Schmidt.

We illustrate the issues with numerical examples. Set the control input function to $b(x) = \exp(1-x)$ throughout and consider two output operators, $C_1 : L^2(0,1) \longrightarrow Y = L^2(0,1)$ and $C_2 : L^2(0,1) \longrightarrow Y = R^1$ defined by $C_1[w(\cdot)] = \sqrt{10}w(\cdot)$ and $C_2[w(\cdot)] = \sqrt{10}\int_0^1 b(\tau)w(\tau)d\tau$, respectively. Here, $Q_1 = C_1^*C_1 = 10I_{L^2}$ is not compact and $Q_2 = C_2^*C_2$ is the Hilbert-Schmidt operator defined by

$$Q_2[w(\cdot)](x) = 10b(x)\int_0^1 b(\tau)w(\tau)d\tau.$$

1) Convergence of Functional Gains: Consider the problem with $\mu = 1/850$, $\nu = 1$ and $Q = Q_1$. In Figure 1 below we compare functional gains computed with the upwind and finite element methods. Note the oscillations in the functional gain computed using the standard finite element scheme. The stabilization parameter was set to be $\alpha = 1/6N$ and the results is typical to what is seen when upwinding is used. However, the stabilization parameter can greatly impact the convergence rate (see [12]).

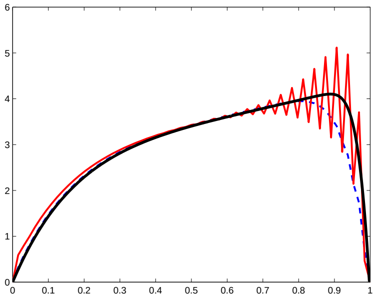

Fig. 1. The red jagged curve is the plot of $k^N(s)$ computed with standard finite elements. The blue dashed curve is the plot of $k^N(s)$ computed with the upwind scheme and the solid black curve is the converged gain.

2) Mesh Independence for the Heat Equation: Consider the system with $\mu = 1/850$ and $\nu = 0$. We investigate mesh independence of the Kleinman-Newton iterations using $Q = Q_1$ (not compact) and $Q = Q_2$ (Hilbert-Schmidt) respectively. The Kleinman-Newton iterations is started with initial operator $\Pi_0 = 2I_{L^2(0,1)}$ and the tolerance is set to be $\|\Pi_k^N - \Pi_\infty^N\| < 10^{-12} = \varepsilon$. As defined in (25), $M^N(\varepsilon, P^N\Pi_0)$ denotes the number of iterations necessary to meet the required tolerance. For briefness we denote $M^N(\varepsilon, P^N\Pi_0)$ by $M_{Q_1}^N$ for the case where $Q = Q_1$ and $M_{Q_2}^N$ when $Q = Q_2$.

Since Q_2 is Hilbert-Schmidt, mesh independence of the Kleinman-Newton iterations is guaranteed, see ([4]). From Table I we see that even though $M_{Q_1}^N$ and $M_{Q_2}^N$ are close, mesh independence is not obtained for the case where Q is not compact.

3) Mesh Independence of the convection diffusion equation and conditioning of the Riccati equation: We consider the convection diffusion equation with $\mu = 1/1500$, $\nu = 1$ and as before, $b(x) = e^{1-x}$. The results obtained using the standard finite element scheme are compared to the upwind

	$Q = Q_1$	$Q = Q_2$
N	$M^N_{Q_1}$	$M^N_{Q_2}$
8	7	7
16	7	7
32	7	6
64	7	6
128	7	6
256	7	6

TABLE I

NUMBER OF ITERATIONS NECESSARY TO SATISFY THE TOLERANCE $\|\Pi_k^N - \Pi_\infty^N\| < 10^{-12}$ FOR $Q = Q_1$ AND $Q = Q_2$ RESPECTIVELY.

scheme for $Q = Q_1$ and $Q = Q_2$ respectively. In addition to comparing the number of iterations, $M^N(\varepsilon, P^N\Pi_0)$, we also compute a lower bound, L, and an upper bound, U, on the condition number of the Riccati equation, see [11].

Let $\Pi_0 = 2I_{L^2(0,1)},\ \varepsilon = 10^{-12}$ and $\alpha = 1/N$.

	FEM	Upwind	FEM		Upwind	
N	$M^N_{Q_1}$	$M^N_{Q_1}$	L	U	L	U
8	7	7	25	79	3	10
16	7	7	13	54	10	37
32	7	7	15	64	27	104
64	7	7	26	112	68	266
128	7	7	51	221	165	656
256	7	7	119	514	402	1645

TABLE II

NUMBER OF KLEINMAN-NEWTON ITERATIONS AND UPPER AND LOWER BOUNDS ON RICCATI CONDITION NUMBER WITH $Q = Q_1$ FOR THE STANDARD AS WELL AS THE UPWIND SCHEME.

	FEM	Upwind	FEM		Upwind	
N	$M^N_{Q_2}$	$M^N_{Q_2}$	L	U	L	U
8	7	7	40	135	1	5
16	7	7	8	32	2	9
32	7	7	4	16	4	16
64	7	7	6	23	10	41
128	6	6	10	42	25	106
256	6	6	21	93	64	275

TABLE III

NUMBER OF KLEINMAN-NEWTON ITERATIONS AND UPPER AND LOWER BOUNDS ON RICCATI CONDITION NUMBER WITH $Q = Q_2$ FOR THE STANDARD AS WELL AS THE UPWIND SCHEME.

Tables II and III illustrate that both schemes behave similarly with respect to the number of iterations necessary to satisfy the given tolerance, regardless of the properties of "Q". If $Q = Q_2$, Hilbert-Schmidt, the bounds on the condition number of the Riccati equation implies that the condition number, for a given N, will be significantly smaller for both schemes than for the case where $Q = Q_1$ is not compact. In addition, the bounds for the standard and upwind schemes are of the same order if $Q = Q_2$, unlike the case where $Q = Q_1$.

VII. CONCLUSIONS

In this short paper we present results concerning mesh independence and numerical conditioning for approximation of infinite dimensional Riccati equations that arise in feedback control of PDE systems of diffusion-convection type. Observe that compactness of the input and controlled output operators is an important factor in convergence and mesh independence. Also, numerical results alone can be misleading unless there is some theoretical foundation to support the results. For example, iteration counts for the discrete problems is not sufficient to ensure mesh independence.

For convection dominated problems, it is possible to establish convergence and mesh independence for finite element and stabilized (upwind) schemes as long as the B and C operators are compact. The numerical conditioning of the Riccati equations is better for the upwind scheme on coarse grids, but on fine grids the finite element method is better conditioned. Mesh independence for the case where C is not compact is not clear.

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