A Distributed Parameter Control Approach to Optimal Filtering and Smoothing with Mobile Sensor Networks

J. A. Burns, Fellow, IEEE, E. M. Cliff, Member, IEEE, and C. Rautenberg
Center for Optimal Design and Control
Interdisciplinary Center for Applied Mathematics
Virginia Polytechnic Institute and State University
Blacksburg, VA 24061-0531.

Abstract

In this paper we present a framework to address filtering and smoothing problems for distributed parameter systems when mobile (dynamic) sensors are used to provide system measurements. This framework can be used for systems governed by parabolic and hyperbolic partial differential equations and hence has application to a diverse set of problems such as estimating locations of biological and chemical sources, target tracking and estimation. We formulate the problems as hybrid systems on infinite dimensional spaces (coupled systems of partial, ordinary and delay differential equations) and use infinite dimensional theory to develop computational algorithms for the problems. A simple numerical example illustrates the approach.

I. Introduction

The background for this framework goes back to the early 1970's when people first started to think about optimal sensor/actuator location problems for distributed parameter systems [7], [10], [20], [23], [25], [26] . Much of the initial research on mobile sensors and actuators focused on achieving more practical observability and controllability conditions. For example, a "typical" controllability condition for the 1D heat equation required that a stationary point actuator be placed at an irrational point in the domain to achieve controllability (see [11]). Clearly this type of result has limited practical application. In 1973 Dolecki [10] first noted that a simple mobile actuator could yield controllability for the 1D heat equation and since then considerable work has been done on mobile control of distributed parameter systems. During the mid 1990's Khapalov produced a series of papers on the design of optimal mobile sensors for a robust filtering problem and applied his results to parabolic and hyperbolic systems (see [15], [16], [17], [18] and [19]).

The problem of optimally placing fixed actuators and sensors to achieve "maximal" controllability or observability of a distributed parameter system is fundamental to estimation and control of such systems. However, the terms "maximal controllability" and "maximal observability" are not always precisely defined, even for finite dimensional systems. Moreover, when the dynamical system in governed by a partial differential equation or a system with delays, the controllability and observability and feedback "gains" are kernel functions in integral representations of feedback operators. We shall consider the case where these operators are computed by solving Riccati equations arising from infinite dimensional estimation and control problems. When these gains exist and can be computed, one has information that provides insight into sensor location and design of low order-local dynamic compensators.

In [3], [5] Burns and King showed that distributed parameter systems described by certain parabolic partial differential equations often have a special structure that smoothe solutions of the corresponding Riccati equation. When this result is applied to problems with distributed controllers it can be established that the resulting feedback operator is also smooth. Both properties are important in addressing sensor and actuator location problems (see [4]) and they have practical implications in the design of reduced order controllers for PDE systems (see [6]). We use these results as a starting point for optimal management of mobile sensor networks and the development of practical computational algorithms.

II. Problem Statement

In order to keep the presentation short we shall limit our discussion to a two dimensional parabolic boundary control problem. Consider a convection/diffusion process in the region $\Omega = [0,1] \times [0,1] \subseteq R^2$ with boundary Γ. The system is described by the partial differential equation with disturbance $v(t)$ given by

$$\frac{\partial}{\partial t}T(t,x,y) = \varepsilon\left[\frac{\partial^2 T(t,x,y)}{\partial x^2} + \frac{\partial^2 T(t,x,y)}{\partial y^2}\right] + a^x(x,y)\frac{\partial T(t,x,y)}{\partial x} + a^y(x,y)\frac{\partial T(t,x,y)}{\partial y} + \sum_{k=1}^{m} b_k(x,y)v_k(t), \quad (1)$$

with boundary condition

$$T(t,x,y)\mid_{\Gamma} = 0 \quad (2)$$

and initial data

$$T(0,x,y) = T_0(x,y). \quad (3)$$

The natural state space for the process is $Z = L_2(\Omega)$ and we define the operator A on the domain

$$\mathcal{D}(A) = H^2(\Omega) \cap H_0^1(\Omega), \quad (4)$$

by

$$A\varphi(\cdot) = \varepsilon\Delta\varphi(\cdot) + \langle \begin{pmatrix} a^x(\cdot) \\ a^y(\cdot) \end{pmatrix}, \nabla\varphi(\cdot)\rangle = \varepsilon\left[\frac{\partial^2\varphi(\cdot)}{\partial x^2} + \frac{\partial^2\varphi(\cdot)}{\partial y^2}\right] + \left[a^x(\cdot)\frac{\partial\varphi(\cdot)}{\partial x} + a^y(\cdot)\frac{\partial\varphi(\cdot)}{\partial y}\right]. \quad (5)$$

Also, we assume the disturbance input functions $b_k(x,y)$ are in $Z = L_2(\Omega)$ so that in input operator $B : R^m \longrightarrow L_2(\Omega)$ defined by

$$B\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_m \end{bmatrix} = \sum_{k=1}^{m} b_k(x,y)v_k \quad (6)$$

is compact.

Assume that one has p sensor-platforms (vehicles) moving in Ω, each with a sensor capable of measuring an average value of $T(t,x,y)$ within an fixed range of the location of the platform. Let $\vec{x}_i(t) = [x_i(t), y_i(t)]^T \in \Omega$, $i = 1,2,...,p$ denote the position of the i^{th} sensor and let

$$h_i(t) = \int\int_{B_\delta(\vec{x}_i(t))} k(x,y)T(t,x,y)\,\mathrm{d}x\,\mathrm{d}y + \eta_i(t)\ . \quad (7)$$

denote the measured output which is the weighted average of the field $T(t,x,y)$ with weight $k(x,y)$ and sensor range

$$B_\delta(\vec{x}_i(t)) \triangleq \left\{\vec{x} \in R^2 : \|\vec{x} - \vec{x}_i(t)\| < \delta\right\}. \quad (8)$$

η is a zero-mean *white* noise process and is uncorrelated with the process disturbance v. Observe that one could also define a dynamic local sensor by

$$h_i(t) = \int\int_{\Omega} \chi(\vec{x}, \vec{x}_i(t))T(t,\vec{x})\,\mathrm{d}\vec{x} + \eta_i(t), \quad (9)$$

where $\chi(\vec{x}, \vec{x}(t))$ is a (normalized) characteristic function defined by

$$\chi(\vec{x}, \vec{x}(t)) = \begin{cases} 1/(\pi\delta^2), & \vec{x} \in B_\delta(\vec{x}(t)) \cap \Omega \\ 0, & \vec{x} \notin B_\delta(\vec{x}(t)) \cap \Omega \end{cases}. \quad (10)$$

This is the definition used by Khapalov (see [15], [16], [17], [18] and [19]) and offers a certain structure that allows for rigorous analysis when the dynamics of the vehicle network is included. For any given network of vehicle trajectories $\vec{x}_i(t) = [x_i(t), y_i(t)]^T \in \Omega$, $i = 1,2,...,p$, we define the output map $C(t) : L_2(\Omega) \longrightarrow R^p$ by

$$C(t)\varphi(\cdot) = \begin{bmatrix} C_1(t)\varphi(\cdot) \\ C_2(t)\varphi(\cdot) \\ C_3(t)\varphi(\cdot) \\ \vdots \\ C_p(t)\varphi(\cdot) \end{bmatrix} \in R^p \quad (11)$$

where

$$C_i(t)\varphi(\cdot) \triangleq \int\int_{\Omega} \chi(\vec{x}, \vec{x}_i(t))\varphi(\vec{x})\,\mathrm{d}\vec{x}. \quad (12)$$

Now one can formulate an abstract (infinite dimensional) model of the form

$$\dot{z}(t) = Az(t) + Bv(t) \in Z \quad (13)$$

with measured output

$$h(t) = C(t)z(t) + \eta(t), \quad (14)$$

where the state of the distributed parameter system is $z(t)(\cdot) = T(t,\cdot) \in Z = L_2(\Omega)$.

One approach to optimal estimation is to observe that the variance equation is an infinite dimensional Riccati (partial) differential equation of the form

$$\dot{\Sigma}(t) = A\Sigma(t) + \Sigma(t)A^* + BB^* - \Sigma(t)C^*(t)C(t)\Sigma(t)\ , \quad \Sigma(t_0) = \Sigma_0\ , \quad (15)$$

and to formulate the sensor management problem as a control problem for (15). In particular, consider the distributed parameter optimal control problem of finding $C_{opt}(t)$ to minimize [23]

$$J(C(\cdot)) = \int_{t_0}^{t_1} \mathrm{tr}\ Q(t)\ \Sigma(t)\ \mathrm{d}t \quad (16)$$

where $\Sigma(\cdot)$ is a mild solution of (15), $C(\cdot)$ is defined by (11)-(12), and for each $t \in [t_0, t_f]$ $Q(t) : L_2(\Omega) \mapsto L_2(\Omega)$ is a bounded linear operator. The (time-varying) map Q allows one to weight significant parts of the state estimate. For example, optimal feedback control may be given by a feedback operator $G : Z \mapsto I\!R^m$. If the re-constructed state is to be used in a feedback controller then one might choose $Q = G^*G$, in effect minimizing the error in the control produced by variance in the state estimate.

If $\tilde{w}(t)$ denotes the vector of sensor-platform positions

$$\tilde{w}(t) = \begin{bmatrix} \vec{x}_1(t) \\ \vec{x}_2(t) \\ \vdots \\ \vec{x}_p(t) \end{bmatrix},$$

then

$$C(t)\varphi(\cdot) = \tilde{C}(\tilde{w}(t))\varphi(\cdot) = \int\int_\Omega \tilde{\chi}(\vec{x}, \tilde{w}(t))\varphi(\vec{x})\ \mathrm{d}\vec{x},$$

where

$$\tilde{\chi}(\vec{x}, \tilde{w}(t)) = \begin{bmatrix} \chi(\vec{x}, \vec{x}_1(t)) \\ \chi(\vec{x}, \vec{x}_2(t)) \\ \vdots \\ \chi(\vec{x}, \vec{x}_p(t)) \end{bmatrix}.$$

The sensor management problem becomes:

> **Problem**$(\mathcal{P})$: Find $\tilde{w}_{opt}(\cdot)$ so that $C(\tilde{w}_{opt}(\cdot))$ minimizes
>
> $$\mathcal{J}(\tilde{w}(\cdot)) = J(\tilde{C}(\tilde{w}_{opt}(\cdot))) = \int_{t_0}^{t_1} \mathrm{tr}\ Q(t)\Sigma(t)\ \mathrm{d}t\ . \tag{17}$$
>
> subject to the constraint (15).

There are several technical and computational challenges that must be addressed in order to solve **Problem**$(\mathcal{P})$ above. We cite the following issues:

(1) Since the variance equation is infinite dimensional, one must be able to establish that the operator $\Sigma(t)$ is of trace class so that the cost functional (16) is well defined over the interval $[t_0, t_1]$. This can be a nontrival problem, but the results in [8], [9], [12], [22], and [24] provide a background to develop the necessary structure.

(2) The numerical solution to the problem requires the introduction of approximations and these numerical algorithms must be developed so that convergence of the schemes is assured. The basic theory and approximation schemes in [4], [6], [9], [12], [14], [22], and [24] may be useful in this task.

(3) The sensor-platform dynamics, geometric constraints and network constraints may be required in certain settings. The leads to hybrid systems that are multi-scale in time and space and may contain communication delays.

III. Theoretical results

We will prove the technical details necessary for **Problem**$(\mathcal{P})$ to be addressed when the sensors are moving along trajectories determined by a controlled differential equation of the form $\dot{\vec{x}} = f(t, \vec{x}, u)$.

Let $\mathscr{I}_1$ and $\mathscr{I}_2$ denote the vector spaces of trace class (nuclear) operators and Hilbert-Schmidt operators respectively, over the same separable complex Hilbert space $\mathscr{H}$. Let $\mathscr{L}(\mathscr{H})$ be the space of bounded linear operators from $\mathscr{H}$ to $\mathscr{H}$, and by $\|A\|$ we write the usual operator norm of A if $A \in \mathscr{L}(\mathscr{H})$. We denote the trace norm of $A \in \mathscr{I}_1$ as $\|A\|_1$, this is equal to what we will define as the trace of $|A|$, i.e.,

$$\|A\|_1 = \mathrm{Tr}\,(|A|) := \sum_{n=1}^{\infty} \langle \phi_n, |A|\phi_n \rangle,$$

for some orthonormal basis $\{\phi_n\}_{n=1}^{\infty}$ of $\mathscr{H}$, where $|A|$ is defined to be the unique positive operator such that $A = U|A|$, where U is a partial isometry. A bounded operator A is a Hilbert-Schmidt operator if $\mathrm{Tr}\,(A^*A) < \infty$ and its norm is defined as $\|A\|_2 = \sqrt{\mathrm{Tr}\,(A^*A)}$. The pairs $(\mathscr{I}_i, \|\cdot\|_i)$ are complete vector spaces for $i = 1, 2$. If $A \in \mathscr{I}_1$, then $\|A\| \le \|A\|_2 \le \|A\|_1$.

Define the spaces $\mathscr{C}([0,\tau]; \mathscr{I}_i)$ for $i = 1, 2$, as

$$\mathscr{C}([0,\tau]; \mathscr{I}_i) = \Big\{F : [0,\tau] \mapsto \mathscr{I}_i \big/ t \mapsto F(t) \text{ is continuous in } \|\ \|_i\Big\},$$

which endowed respectively with the norms $|||F|||_i = \sup_{t\in[0,\tau]} \|F(t)\|_i$ are Banach spaces. We will prove now that the solution of the Riccati integral version of (14) is of trace class for each t.

Theorem 1: Let $T(t)$ be a C_0 semigroup on $\mathscr{H}$, $\Sigma_0 \in \mathscr{I}_1$ and $\Sigma_0 \ge 0$, $B \in \mathscr{I}_2$, and $C(\cdot) \in \mathscr{C}([0,\tau]; \mathscr{I}_2)$, then the equation $\Sigma = \gamma(\Sigma)$ has a unique solution in $\mathscr{C}([0,\tau]; \mathscr{I}_1)$, where $\gamma : \mathscr{C}([0,\tau]; \mathscr{I}_1) \mapsto \mathscr{C}([0,\tau]; \mathscr{I}_1)$ is defined by

$$\gamma(\Sigma)(t) = T^*(t)\Sigma_0 T(t) + \int_0^t T^*(t-s)(BB^* - \Sigma(s)(C^*C)(s)\Sigma(s))T(t-s)\ \mathrm{d}s. \tag{18}$$

Proof. Since $B \in \mathscr{I}_2$ and $C(\cdot) \in \mathscr{C}([0,\tau]; \mathscr{I}_2)$, then $BB^* \in \mathscr{I}_1$ and $C^*C(\cdot) \in \mathscr{C}([0,\tau]; \mathscr{I}_1)$, then the integral term in the definition of γ belongs to $\mathscr{C}([0,\tau]; \mathscr{I}_1)$ if $\Sigma(\cdot) \in \mathscr{C}([0,\tau]; \mathscr{I}_1)$. Also since $\Sigma_0 \in \mathscr{I}_1$ then $T^*(\cdot)\Sigma_0 T(\cdot)$ belongs to $\mathscr{C}([0,\tau]; \mathscr{I}_1)$ and then γ is defined from $\mathscr{C}([0,\tau]; \mathscr{I}_1)$ to the same space.

The proof requires a slight modification of the one of Bensoussan and Da Prato in [2]. Since $T(t)$ is a C_0 semigroup, there is a M_τ such that $\|T(t)\| \le M_\tau$ for all $t \in [0,\tau]$, then define $\beta = M_\tau^2(\|\Sigma_0\|_1 + \tau\|B\|_2^2)$ and pick ρ and $s \le \tau$ such that

$$\rho = 2M_\tau^2\beta; \qquad s(\|B\|_2^2 + \rho^2|||C|||_2^2) \le \beta; \qquad 2\rho s M_\tau^2 |||C|||_2^2 \le \frac{1}{2};$$

then the mapping γ defines a contraction on the ball

$$B_{s,\rho} = \{F \in \mathscr{C}([0,s];\mathscr{I}_1) / |||F|||_1 \leq \rho\},$$

and then the equation $\Sigma = \gamma(\Sigma)$ defines an unique solution on $B_{s,\rho}$. Since $\Sigma(t) \geq 0$ (see [8]) for $t \in [0,s]$, and $\mathrm{Tr}\,(\cdot)$ is a bounded linear functional on $\mathscr{I}_1$, then we observe that $\|\Sigma(t)\|_1 \leq M_\tau^2(\|\Sigma_0\|_1 + s\|B\|_2^2) \leq \beta$, on $[0,s]$ and this allows to repeat the contraction argument on the interval $[s,2s]$ and so on.

■

We can now prove that the minimization of the functional in (16) is well defined over a compact set of operators in $\mathscr{C}([0,\tau];\mathscr{I}_2)$. We know that $\mathrm{Tr}\,(\cdot)$ defines a linear functional on $\mathscr{I}_1$, and $\Sigma(\cdot) \mapsto \int_0^\tau \mathrm{Tr}\,(Q\Sigma)\,(t)\,\mathrm{d}t$ is linear and also bounded on $\mathscr{C}([0,\tau];\mathscr{I}_1)$ as $|\int_0^\tau \mathrm{Tr}\,(Q\Sigma)\,(t)\mathrm{d}t| \leq \tau \sup_{t\in[0,\tau]} \|Q(t)\|\; |||\Sigma|||_1$ and then the following proof is equivalent on proving continuity of the mapping $C(\cdot) \mapsto \Sigma_C(\cdot)$, where $\Sigma_C(\cdot)$ is the solution of the integral Riccati equation for that particular $C(\cdot)$.

Theorem 2: Assume all the hypothesis of Theorem 1. Let $\mathcal{F}$ be a compact subset of $\mathscr{C}([0,\tau];\mathscr{I}_2)$, $Q(t) \geq 0$ and $t \mapsto Q(t)$ continuous for each $t \in [0,\tau]$, then

$$J(C(\cdot)) = \inf_{\hat{C}\in\mathcal{F}} J(\hat{C}(\cdot)) \geq 0,$$

for some $C(\cdot) \in \mathcal{F}$, where J is defined by 16.

Proof. Denote $\Sigma_C(\cdot)$ to the solution of the Riccati equation for a particular $C(\cdot) \in \mathcal{F}$. Since $\Sigma_C(t) \geq 0$ and belongs to $\mathscr{I}_1$ for each t and we have chosen $Q(t) \geq 0$, then $\mathrm{Tr}\,(Q(t)\Sigma_C(t)) \geq 0$ (see [13]), therefore $J(C) = \int_0^\tau \mathrm{Tr}\,(Q\Sigma_C)\,(t)\,\mathrm{d}t$, is bounded below by 0.

By compactness we have a convergent minimizing sequence $C_n(\cdot)$ in $\mathcal{F}$ to some $C(\cdot)$ in the $\mathscr{C}([0,\tau];\mathscr{I}_2)$ norm, by properties of the norms in $\mathscr{I}_1$ and $\mathscr{I}_2$ this implies that $C_n^*C_n(\cdot) \to C^*C(\cdot)$ in the $\mathscr{C}([0,\tau];\mathscr{I}_1)$ norm.

Since $\mathcal{F}$ is compact, then there is a $c > 0$ such that $|||C_n|||_2 \leq c$ for all n, then we can choose one time interval $[0,s]$ such that the the mappings γ_C and γ_{C_n} (defined to be the mappings γ in Equation (18) with $C(\cdot)$ and $C_n(\cdot)$ respectively) are all $\frac{1}{2}$ contractions on some ball $B_{s,\rho}$. Without loss of generality suppose that $s = \tau$ (if $s < \tau$, the following argument can be used a finite number of times over the intervals $[s,2s]$, $[2s,3s]$, etc).

Let $\{\Sigma_C^m(\cdot)\}_{m=1}^\infty$ and $\{\Sigma_{C_n}^m(\cdot)\}_{m=1}^\infty$ denote the usual recurring sequences determined by the contraction mapping principle $\Sigma_C^m(\cdot) = \gamma(\Sigma_C^{m-1}(\cdot))$ and $\Sigma_{C_n}^m(\cdot) = \gamma_n(\Sigma_{C_n}^{m-1}(\cdot))$ converging to $\Sigma_C(\cdot)$ and $\Sigma_{C_n}(\cdot)$ respectively and with $\Sigma_C^1(\cdot) = \Sigma_{C_n}^1(\cdot) = 0$. Since $C_n^*C_n(\cdot) \to C^*C(\cdot)$ in the $\mathscr{C}([0,\tau];\mathscr{I}_1)$ topology, it can be proven inductively that $\lim_{n\to\infty} \Sigma_{C_n}^m(\cdot) = \Sigma_C^m(\cdot)$, for each $m = 1,2,\ldots$.

Then,

$$\Sigma_C - \Sigma_{C_n} = \left(\Sigma_C - \Sigma_C^m\right) + \left(\Sigma_{C_n}^m - \Sigma_{C_n}\right) + \left(\Sigma_C^m - \Sigma_{C_n}^m\right)$$

the first term in parenthesis goes to zero as $m \to \infty$ by the contraction mapping Theorem, the second term by assumptions of the mapping γ_n, verifies $|||\Sigma_{C_n}^m - \Sigma_{C_n}|||_1 \leq \left(\frac{1}{2}\right)^m |||\Sigma_{C_n}|||_1 \leq \left(\frac{1}{2}\right)^m \rho$ so it also goes to zero as $m \to \infty$ uniformly in n and the last term in parenthesis goes to zero as $n \to \infty$, therefore $\Sigma_{C_n}(\cdot) \to \Sigma_C(\cdot)$ in the $\mathscr{C}([0,\tau];\mathscr{I}_1)$ topology.

Finally, $\mathrm{Tr}\,(\Sigma_{C_n}(t)Q(t)) \to \mathrm{Tr}\,(\Sigma_C(t)Q(t))$ uniformly in t since $|\mathrm{Tr}\,(\Sigma_C(t)Q(t)) - \mathrm{Tr}\,(\Sigma_{C_n}(t)Q(t))\,| \leq q|||\Sigma_{C_n} - \Sigma_C|||_1$, with $q = \sup \|Q(t)\|$, therefore

$$\inf_{\hat{C}\in\mathcal{F}} J(\hat{C}(\cdot)) = \lim_{n\to\infty} \int_{t_0}^{t_1} \mathrm{Tr}\,(Q(t)\Sigma_{C_n}(t))$$
$$= \int_{t_0}^{t_1} \mathrm{Tr}\,(Q(t)\Sigma_C(t)) = J(C(\cdot)).$$

■

We will prove that the set of operators of the form (12), determined by the trajectories, which are solutions to some nonlinear ODE, of one moving sensor is compact under certain assumptions, but the proof can be extended to finite number of moving sensors.

Let $(x(t),y(t)) = \vec{x}(t) = \vec{x}(t,\vec{x}_0,u)$ be such that $\vec{x}(\cdot,\vec{x}_0,u) : [0,\tau] \mapsto \Omega \equiv [0,1]\times[0,1]$ is a solution of

$$\dot{\vec{x}} = f(t,\vec{x},u), \qquad (19)$$
$$\vec{x}(0) = \vec{x}_0, \qquad (20)$$

with $f \in C^1$, $\vec{x}_0 \in \Omega$ fixed and $u \in \mathcal{U}$, where

$$\mathcal{U} = \{u/u \text{ is measurable and } u(t) \in \Gamma \text{ for all } t \in [0,\tau]\},$$

and Γ is compact. We shall suppose the following

a) The response verifies $\vec{x}(t,\vec{x}_0,u) \in \Omega$, for all $u \in \mathcal{U}$ and all $t \in [0,\tau]$, so $\vec{x}(t,\vec{x}_0,u)$ is uniformly bounded.

b) The set $V(\vec{x},t) = \{f(t,\vec{x},u)/u \in \Gamma\}$ is convex for each fixed $(\vec{x},t)$.

For the case of one moving sensor, the operator $C(t)$ in (11) is defined as $C(t) : L^2(\Omega) \mapsto I\!R$, but it can also defined as $C(t) : L^2(\Omega) \mapsto L^2(\Omega)$ by simply assigning the value $C(t)\varphi(\cdot)$ to be a constant function on Ω. Similarly, in the case of p moving sensors, we might define $(C(t)\varphi)(\vec{x}) = \sum_{i=1}^p \chi(\vec{x},\vec{x}_i(t))C_i(t)\varphi(\vec{x})$ where the $C_i(t)$ operators are defined in (12) as $C_i(t) : L^2(\Omega) \mapsto I\!R$, so that $C(t) : L^2(\Omega) \mapsto L^2(\Omega)$.

Let $\mathcal{F}$ be the set of all operators $C(t) : L^2(\Omega) \mapsto L^2(\Omega)$ of the form

$$C(t)\varphi(\cdot) = \int_\Omega \chi(\vec{x},\vec{x}(t))\varphi(\vec{x})\,\mathrm{d}\vec{x}, \qquad (21)$$

where $\vec{x}(t) = \vec{x}(t,\vec{x}_0,u)$ is the solution of the differential equation (19) with initial condition (20).

Theorem 3: The set $\mathcal{F}$ is compact in the $\mathscr{C}([0,\tau];\mathscr{I}_2)$ topology.

Proof. Let $C(\cdot) \in \mathcal{F}$, then since for each $t \in [0,\tau]$, the kernel of the integral representation for each $C(t)$ is square integrable in the Lebesgue measure on $\Omega \times \Omega$, the operator $C(t)$ is Hilbert-Schmidt. Since the trajectory $\vec{x}(\cdot)$ is continuous in t for each $u \in \mathcal{U}$, then $t \mapsto C(t)$ is continuous in the Hilbert-Schmidt norm because $\|C(t_1) - C(t_2)\|_2^2 = \int_{\Omega\times\Omega} |\chi(\vec{x},\vec{x}(t_1)) - \chi(\vec{x},\vec{x}(t_2))|^2 \,\mathrm{d}\vec{x}\,\mathrm{d}\vec{y} \leq m\|\vec{x}(t_1) - \vec{x}(t_2)\|_{I\!R^2}$ for some $m > 0$. Therefore, for each $u \in \mathcal{U}$, we observe $C(\cdot) \in \mathscr{C}([0,\tau];\mathscr{I}_2)$.

Following the proof of Theorem 2, Chapter 4 of Lee and Markus in [21], consider a sequence of controls $u_i \in \mathcal{U}$, then the sequence of functions $\int_0^t f(t,\vec{x}_i(s),u_i(s))\,\mathrm{d}s$ is uniformly bounded and equicontinuous, then by the Arzela−Ascoli Theorem there is a subsequence of u_i (that we call again u_i) such that $\int_0^t f(t,\vec{x}_i(s),u_i(s))\,\mathrm{d}s \to \int_0^t \bar{\phi}(s)\,\mathrm{d}s$ uniformly in $[0,\tau]$ for some integrable function $\bar{\phi}$. By the proof we've mentioned, there is a $\bar{u} \in \mathcal{U}$ such that $\bar{\phi}(t) = f(t,\bar{x}(t),\bar{u}(t))$ for $t \in [0,\tau]$.

Let then C_i be the sequence of operators defined by $\vec{x}_i(\cdot,\vec{x}_0,u_i)$ and $\bar{C}$ be the one defined by $\bar{x}(\cdot,\vec{x}_0,\bar{u})$, then $\|C_i(t) - \bar{C}(t)\|_2^2 \leq m\|\vec{x}_i(t) - \bar{x}(t)\|_{I\!R^2}$ for some $m > 0$, since $x_i(t) \to \bar{x}(t)$ uniformly in t, then $C_i \to \bar{C}$ in the $\mathscr{C}([0,\tau];\mathscr{I}_2)$ topology.
■

IV. A Numerical Example

To illustrate the ideas with limited computational burden, we consider a one-dimensional convection/diffusion model:

$$T_t(t,\xi) = \varepsilon T_{\xi\xi}(t,\xi) - a\,T_\xi(t,\xi) + b(x)v(t), \qquad 0 \leq \xi \leq 1, \quad t_0 < t \leq t_1\ , \tag{22}$$

with boundary conditions

$$T_\xi(t,0) = 0, \qquad T_\xi(t,1) = 0\ , \tag{23}$$

and prescribed initial condition

$$T(t_0,\xi) = T_0(\xi). \tag{24}$$

The constant a is given.

Furthermore, we consider a single sensor-platform with output given by

$$h(t) = \frac{1}{2\delta}\int_{x-\delta}^{x+\delta} T(t,\xi)\,\mathrm{d}\xi + \eta(t)\ , \tag{25}$$

where the sensor location is given by

$$x(t) = \frac{(t-t_0)\,x_f + (t_f - t)\,x_0}{(t_f - t_0)}\ . \tag{26}$$

Thus, the sensor moves from $\xi = x_0$ at time $t = t_0$ to $\xi = x_f$ at $t = t_f$. Since the sensor path is parameterized by $(x_0,x_f) \in I\!R^2$, it's clear the cost functional in (17) depends on these two parameters (*i.e.* $\mathcal{J}(x_0,x_f)$. To evaluate this functional we introduce approximations for the system (abstractly (13), (14)), and for the Riccati differential equation (15).

Given an integer $n > 2$ we introduce a uniform grid of $n+1$ points on the spatial interval $[0,1]$, and consider the continuous, piecewise linear (*hat*) functions on this grid with $\phi_\imath^n(t_\jmath) = \delta_{\imath\,\jmath}$. In this setting the dynamics (22 - 23-24), projected to the $(n{+}1)$-dimensional space (Z^n) spanned by $\{\phi_1^n,...,\phi_{n+1}^n\}$, may be represented by

$$M^n\dot{z}^n(t) = K^n z^n(t) + B^n v(t)\ .$$

To approximate the sensor output (25) we introduce a $(1\times n{+}1)$ output matrix (C^n) with

$$C_\jmath^n(t) = \frac{1}{2\delta}\int_{x(t)-\delta}^{x(t)+\delta} \phi_\jmath^n(\xi)\,\mathrm{d}\xi, \;\; \text{so that}\ .$$

so that $h^n(t) \sim C^n(t)\; z^n(t)$. Approximation for the Riccati pde (15) follows from approximations of the operators A,B,C, although care must be taken in representing the various adjoint operators.

These calculations were implemented for the case:

$$\varepsilon = 0.01,\ a = 0.80,\ \delta = 0.05,\ t_0{=}0,\ t_f{=}.2,\ n = 128.$$

We take ${\Sigma_0}^n = (M^n)^{-1}$.

The projected Riccati differential equation (initial value problem) was solved on $[0,0.2]$ using the MATLAB procedure **ode23**, and the cost functional was evaluated using trapezoidal integration. Symmetry in the Riccati matrix was exploited. Note that more efficient Riccati solvers are available [1], but have not yet been implemented here.

Figure 1 displays a surface plot from a survey over the parameter range $0.1 \leq x_0, x_f \leq 0.9$. It appears that there is a *valley* along the line $x_0 + x_f \approx 1$. In Figure 2 we display line plots along the parameter lines $x_0 = x^f$ (red) and $x_0 + x_f = 1$ (blue). Finally, we used **fminconstrained** from MATLAB's **Optimization Toolbox** to find $x_0^* = 0.592,\ x_f^* = 0.590$. It appears that, for this example, the best performance is achieved with a *nearly* stationary sensor.

V. Summary

An approach has been sketched for optimal filtering for a class of distributed parameter systems with mobile sensors. A filtering problem has been formulated wherein the state-to-output map depends on the path of the sensor-platform(s). A cost-functional is defined as the time-integral of the trace of the (weighted) covariance. A 1D convection/diffusion equation with parameterized sensor-platform motion was used to illustrate the ideas. We noted that an optimal stationary sensor performed as well as a mobile sensor for this case. Similar results were observed in 2D problems. Also, when one includes the sensor dynamics, then we expect the mobile sensor network to perform

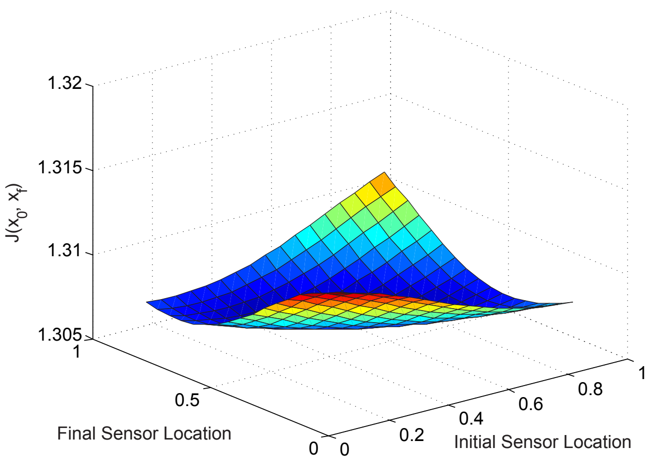

Fig. 1. Cost functional surface, $\varepsilon=.01,\ a=.8,\ \delta=.05,\ t_f=.2,\ n=128$

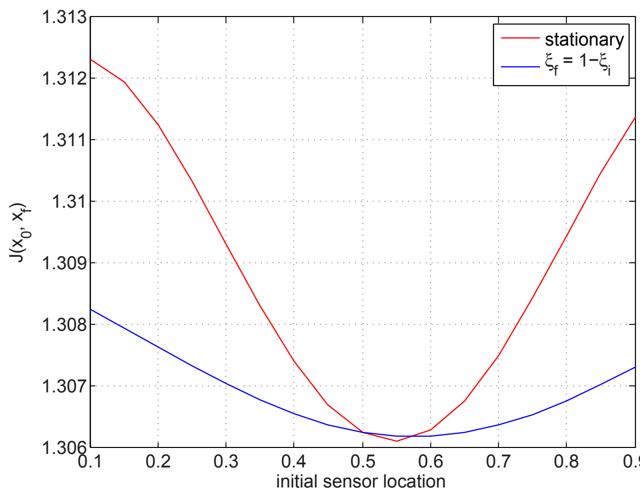

Fig. 2. Line plots, $\varepsilon=.01,\ a=.8,\ \delta=.05,\ t_f=.2,\ n=128$

less effectively. The problem is to determine the optimal performance under dynamics sensor constraints. This is a highly nonlinear, hybrid control problem and presents a huge computational challenge.

VI. Acknowledgement

This work has been supported in part by the Air Force Office of Scientific Research under Grant FA9550-07-1-273.

References

[1] P. Benner and H. Mena, BDF Methods for Large-Scale Differential Riccati Equations, TU Chemnitz, preprint.

[2] A. Bensoussan, G. Da Prato, M. Delfour and S. K. Mitter *Representation and Control of Infinite Dimensional Systems*, Vol. I, II, Birkhäuser, 1992.

[3] J. A. Burns and B. B. King, A Note on the regularity of solutions of infinite dimensional Riccati equations, *Appl. Math. Lett.*, 7 (1994), 13–17.

[4] J. A. Burns and B. B. King, Optimal sensor location for robust control of distributed parameter systems, In *Proc. of the 33rd IEEE Control and Decision Conference*, Dec. 1994, 3967–3972.

[5] J. A. Burns, D. Rubio and B. B. King, Regularity of Feedback Operators for Boundary Control of Thermal Processes, In *Proc. First International Conference On Nonlinear Problems in Aviation and Aerospace*, S. Sivasundaram, Ed., Embry-Riddle Aeronautical Press, May, 1996, 67–73.

[6] J. A. Burns and B. B. King, A reduced basis approach to the design of low order feedback controllers for nonlinear continuous systems, *Journal of Vibration and Control*, 4 (1998), 297–323.

[7] R. Curtain, A Survey of Infinite-Dimensional Filtering, *SIAM Review*, 17 (1975), 395-411.

[8] R. F. Curtain and H. Zwart, *An Introduction to Infinite-Dimensional Linear Systems Theory*, Springer, Berlin, 1995.

[9] A. De Santis, A. Germani and L. Jetto, Approximation of the Algebraic Riccati Equation in the Hilbert Space of Hilbert-Schmidt Operators, *SIAM J. Control Opt.*, 31 (1993), 847–874.

[10] Sz. Dolecki, Observation for the One-dimensional Heat Equation, *Stadia Math.*, 48 (1973), 291–305.

[11] A. El Jai AND A. J. Pritchard, *Sensors and Actuators in the Analysis of Distributed Systems*, John Wiley, New York, 1988.

[12] J. S. Gibson, The Riccati Integral Equations for Optimal Control Problems on Hilbert Spaces, *SIAM J. Control and Opt.*, 17 (1979), 537–565.

[13] I. C. Gohberg and M. G. Kreĭn, *Introduction to the Theory of Linear Nonselfadjoint Operators*, American Mathematical Society, 1969.

[14] K. Ito, Strong convergence and convergence rates of approximating solutions for algebraic Riccati equations in Hilbert spaces, In *Distributed Parameter Systems*, eds. F. Kappel, K. Kunisch, W. Schappacher, Springer-Verlag, (1987), 153–166.

[15] A.Yu. Khapalov, Optimal measurement trajectories for distributed parameter systems, *Systems Control Lett.*, 18 (1992), 467–477.

[16] A.Yu. Khapalov, Observability of hyperbolic systems with interior moving sensors, In *Lecture Notes in Control and Inform. Sci. 185*, R. F. Curtain, A. Bensoussan and J.-L. Lions, Eds., Springer-Verlag, New York, 1993, 489–499.

[17] A.Yu. Khapalov, L^∞-Exact Observability of the Heat Equation with Scanning Pointwise Sensor, *SIAM J. Control and Opt.*, 32 (1994), 1037–1051.

[18] A.Yu. Khapalov, Controllability of the wave equation with moving point control, *Appl. Math. Optim.*, 31 (1995), 155–175.

[19] A.Yu. Khapalov, Exact observability of the time-varying hyperbolic equation with finitely many moving internal observations, *SIAM J. Control and Opt.*, 33 (1995), 1256–1269.

[20] S. Kumar and J.H. Seinfeld, Optimal Location of Measurements for Distributed Parameter Estimation, *IEEE Trans. on Auto. Control* 23 (1978), 690-698.

[21] E. B. Lee and L. Markus, *Foundations of Optimal Control Theory*, John Wiley, 1969.

[22] K. Morris, LQ-Optimal Actuator Location and Norm Convergence of Ricatti Operators, In *Proc. of the 47th IEEE Control and Decision Conference*, Dec.2008, 209–214.

[23] S.Omato, S. Koide, and T Soeda, Optimal Sensor Location Problem for a Linear Distributed Parameter System, *IEEE Trans. on Auto. Control* 23 (1978), 665-673.

[24] I. G. Rosen, On Hilbert-Schmidt Norm Convergence of Galerkin Approximation for Operator Riccati Equations, *International Series of Numerical Mathematics 91*, Birkhäuser, Basel, Switzerland, 1989, 335-349.

[25] Y. Sakawa, Controllability for partial differential equations of parabolic type, *SIAM J. Control*, 12 (1974), 389–400.

[26] Y. Sakawa, Observability and related problems for partial differential equations of parabolic type, *SIAM J. Control*, 13 (1975), 14–27.