A Conversation With Archimedes

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In my history of mathematics class, we were studying Archimedes; the discussions were often intense, and the previous class meeting – concerning his method of dividing a sphere into segments – was no exception. At the end of class, I told them, "Next time, we'll talk about the thirteen Archimedean solids."

I went to my office, thinking about "The Archimedes Codex," a book I had recently read, and unlocked my office door. I was startled to see a stranger standing there, looking at a colorful poster depicting several families of polyhedra: the Platonic and Archimedean Solids, the Prisms and Anti-Prisms, and the Johnson Solids. The stranger was oddly dressed, and I wondered how he had gotten into my office. Then I recognized him, at which point he spoke, and we had this conversation:

The Stranger: This is a beautiful poster, Friend. Did you construct it?

Math Horizons: No, it was a present from a colleague.

Archimedes: A wise colleague, I'm sure.

MH: I'll tell her you said so; and are you, by any chance, Archimedes of Syracuse?

A: Yes, I am.

MH: Sir, you are generally regarded as the greatest scientific mind of antiquity, and mathematicians and students of our time are really curious about you, so may we ask you some questions?

A: Why, certainly.

MH: First of all, what does the name "Archimedes" mean?

A: Oh, the authors of "The Archimedes Codex" discuss my name. It comes from *arche*, meaning "rule", "principle", or "first", and *medos*, meaning "mind" or "wisdom". Now, in my day, "principle" meant something like "foundation of the universe". Now, there's a little confusion about my name. Read it as you do now and it means something like "leading mind"; read it as we Greeks frequently did in those days, and it means "mind of the principle". Take your pick.

MH: Please tell us about your early life.

A: I was born in Syracuse, which was, and still is, a port city located in southern Sicily. In those days, birth dates were only recorded among the nobility and royalty.

My father was merely an astronomer, so I really don't know my exact date of birth. The rulers of Syracuse were called Tyrants, and by the modern sense of the word, some were and some were not. I was probably born during the tyranny of either Icetas or Sosistratus, and I was a youth when my patron Hieron II began his rule. By your reckoning, that was the year 275 B.C.E., so I was born between 290 and 275. The fifty years of Hieron's tyranny were years of peace, and Friend, that was important.

MH: Why so, Archimedes?

A: Think about it, Friend. Fifty years of peace and prosperity under one enlightened ruler meant years of support for artists, musicians, educators, and thinkers. It was my good fortune to live in those times, and to have Heiron's support. He provided for me so that I could spend my time thinking about mathematics and mechanics. We were kinsmen; I was never sure of the exact kinship, but I think his grandmother and my father's uncle were cousins. Who knows? Anyway, my father was Hieron's court astronomer, and they say I was quick to learn to read and write, and even quicker at counting.

MH: Your education?

A: Tutors and family members, at first; then – when I was old enough to travel – it was off to Alexandria, to study with the Masters.

MH: Did you study ...

A: ... with Euclid? Pretty idea, Friend, but he was before my time. The Museon, including the Library, was a remarkable institution, founded about the time I was born by that equally remarkable Egyptian ruler Ptolemy Soter. It was he who brought to Alexandria great thinkers of all stripes, specialties, and origins, supporting them so that they could think about questions of importance to the kingdom of Egypt. During my stay, I learned from the "Elements" all of the mathematics in the Western World: geometry, triangles, Eudoxus' method of exhaustions, numbers, proportions, your Platonic solids up there on the wall ... and especially quadrature — how to describe curved regions or solids as regions or solids bounded by lines or planes. I took a special interest in quadrature, particularly those vexing compass-and-straightedge problems. You know them: the Duplication of the Cube, Trisection of the Angle, and Quadrature of the Circle. Quite maddening!

MH: But you showed how to trisect a general angle with a compass and a marked straightedge, as well as with your spirals, didn't you?

A: Yes, that's right. I loved those spirals, by the way ... such beautiful figures.

Let's talk about them later, shall we?

MH: Certainly. Now, about the construction problems...

A: Right. The importance of those three problems might be overlooked in your time, but let me tell you, Friend, that those three problems began a mathematical tradition that has persisted for two-and-a-half millennia, namely, posing specific questions to be answered in specific ways. Construct a cube equal to twice a given cube, using only compass and straightedge. Find a solution to the general cubic equation that uses algebra and not geometry. Construct a proof of the Prime Number Theorem that does not use complex analysis. Find a primality test that is deterministic and runs in polynomial time — a recent remarkable achievement, I might add. It all began with those construction problems.

Anyway, when I returned to Syracuse, I corresponded with scholars all throughout the Mediterranean world ... especially my good friends Conon of Samos and Alexandria's great mind, Eratosthenes.

MH: Why were your writings practically forgotten, and only came to us by the slenderest of threads, while Euclid's "Elements" achieved enormous popularity in every corner of the scientific world and since the 15th century has gone through more than a thousand printed editions?

A: Euclid's "Elements" contained, at the time, all of the mathematics anyone needed to know in order to be an educated person. Euclid's approach was to begin with common notions and postulates and to use these to build up a body of theorems. He wrote beautifully, and the "Elements" persisted throughout the millennia because it was very much a textbook. You could teach someone mathematics from it. You could teach yourself from it. You still can, and many have done so. Newton supposedly read it and found it "wanting", but his teacher Isaac Barrow pointed out that to understand Descartes properly, Newton should go back and master Euclid — and Newton did so. No, Friend, the "Elements" survived because it is the quintessential mathematics textbook.

On the other hand, I was interested in answering more advanced questions in mathematics and mechanics. I wrote for myself and for a few colleagues who could understand my work. Nowadays, there is an entire mathematics curriculum that leads students from the introductory level (such as Euclid) to the research level (such as me.) With your background, you can read my works, and with a little extra effort and training, your students can read my works as well. Various people rediscovered my writings through the ages and transcribed them, mainly for themselves. They were translated into Arabic by scholars who were knowledgable enough to appreciate them, but apparently some Greek copies survived into the Middle Ages, when ... but you can read some of that story in "The Archimedes Codex". In a word, Euclid treated the fundamentals, and I did research.

MH: Apparently, you have read that new book.

A: The one about the rediscovery of the so-called Archimedes Palimpsest? Yes. Now, your readers might not know that that a palimpsest is a book written on sheets of parchment that have been scrubbed clean of previous writing. The one bearing my name is a thirteenth century prayer book, and the previous writing included several of my treatises. So, you could say that I wrote it, couldn't you?

MH: Hmmm...good point. Your works were copied and recopied during the seventeen centuries between your death and the invention of printing with movable type. In the fifteenth century, the great astronomer and mathematician Regiomontanus praised the latter invention as a way of removing the irritation of copying errors. He did not anticipate typos and misprints. How did your work fare through all of that copying?

A: Bless all of those hardworking scribes down through the ages, especially the ones who were both diligent and mainly ignorant of mathematics!

MH: Ignorance was good?

A: Yes. A scribe who knows mathematics thinks too much about the content of the work in hand, which sometimes leads to such thoughts as "well, he says *this*, but that's impossible, so he must have meant *that*," when I meant no such thing. Worse than that, he might change one of my figures to resemble contemporary mathematical figures, instead of what I drew in the first place. A scribe who knows no mathematics copies exactly what he sees without it going through a filter of prior knowledge, and so the writing has a better chance of being transcribed exactly. Do you see what I mean? And that tenth century scribe, who created the codex that was transformed into what you call the Archimedes Palimpsest, was such a scribe; he preserved my writings with great care and attention to detail.

MH: What about the lack of algebraic symbols in your work?

A: George Pólya always says, "Draw a figure." He's so right. In my day, figures were more like schematic aids to the argument at hand. They were supposed to be what you'd call generic, and you had to be careful not to mislead the reader. For example, if you drew a straight line from one corner A of a triangle to the opposite side BC, the point of intersection should not look like the midpoint of BC unless you meant it to be just that. Figures were vital in all the mathematical works of my day.

MH: The most famous story about your life is “The Crown and the Bath”. Tell us about that.

A: Oh, *that* story! Hieron had commissioned a goldsmith to make a crown — it looked more like a wreath, actually — and gave him the gold to make it. For some reason, he suspected that the goldsmith kept some of the gold and made the crown out of an alloy of gold and some base metals. He asked me to figure out how to determine the composition of the crown without melting it down. That’s the problem.

MH: What about your shouting, “Eureka!” and running naked through the streets when you figured out a solution?

A: Pure silliness. Public baths were places you went to relax, to be sociable, and sometimes just to sit and think. First of all, everyone notices that when you get into a bath, the water level rises. Haven’t you?

MH: Well, yes, but ...

A: ... but if you are curious about the physical world, you just might think about why you float, or what makes sailing vessels float, even those made of metal. For me, the key question was not “Why do I float?” but “What holds me up?”

I got in the bath and displaced some water. That water was held in place exactly by the upward force of the water under it. The volume of water I displaced weighs more than me, so my downward force on the water below was less than the upward force from that water. Consequently, I was held up by that force. Similarly, a hollow metal vessel is mostly air, and so it is also less dense than water. The volume of water it displaces weighs more than it does, and again, the force of the water holds the vessel up. That’s buoyancy, don’t you see?

The point is that while I was thinking about the crown problem, I made this much more important observation about floating bodies. Of course, I was happy. Of course, I shouted “Eureka!” And of course, I did *not* run naked screaming through the streets of Syracuse; that would reflect poorly on the royal house of Hieron and embarrass my family. It’s all in my book “On Floating Bodies”, you know. The science, not the story.

MH: Could you have solved the problem of the crown without that insight?

A: Certainly, Friend. In my day, it was known that volumes of objects of equal mass vary inversely as their densities. A crown of gold is very dense, and a crown of the same weight made of an alloy of gold and baser metal will be less dense. The latter will displace more water that the crown of gold, and a *very* careful experiment will reveal this difference. But I didn't even need to do that.

MH: You didn't need to immerse the crown in water?

A: Not at all. I told one of my boyhood friends in Syracuse, a metal worker, about the crown, and he said, "Make a crown out of pure gold? You jest, Archimedes! Unalloyed gold is too soft a metal for a crown, and thin pure-golden ornaments, such as crowns, are too easily bent and damaged. Just try bending Hieron's crown...gently, of course! If it bends easily, it's gold. If not, it's an alloy." I took his advice and tested the crown. It strongly resisted being bent, and so it was not pure gold.

MH: But why ...

A: ... why the story? I really don't know. The earliest source is Vitruvius, the Roman author who made it up two hundred years later? Was he there? No. Similar stories abound about Newton, Michelangelo, Galileo, Gauss, Mozart, Beethoven, Hilbert, Van Gogh, and Einstein. They become folk heroes, for some reason. Elvis, too: a very polite young man.

MH: Let's go back to the Archimedean solids. According to Pappus, you were the one who first described those semi-regular polyhedra that bear your name; is that right?

A: I certainly wrote a manuscript about them, but was I the first? Who knows? As a student in Alexandria, I studied the Platonic solids, those regular polyhedra in which all faces are congruent regular polygons and the same number of faces meet at each corner. One thing I learned was that Plato knew about the cuboctahedron, the one with six square and eight triangular faces.

Now, the floors of many buildings were covered with square tiles, but in one room a heavy object had fallen and smashed a corner where four tiles met. The repair job made it look as if the corner had been replaced by a small square tile rotated so that its corners lined up with the edges of the original tiles. I saw that if each corner were similarly replaced, and if all the edges were the same length, then the result would be a new kind of tiling. This would have two regular octagons and one square meeting at each corner, and that would be interesting.

Many years later, I remembered this incident, went to my workshop and sliced off all eight corners of a cube. With care, this truncated cube would be a new kind of solid, with 24 corners, 36 edges, six regular octagonal faces, and eight equilateral triangular faces. Two octagons and a triangle would meet at each corner. With a little more slicing, the triangles expanded, the octagons became squares, and I constructed the cuboctahedron: the one Plato knew. This was really interesting! Now I got serious about it and was able to work out the shapes of all possible polyhedra with regular polygonal faces having the same configuration of faces at each corner, and having more than one kind of face. I wrote up an account of these solids; Heron of Alexandria mentioned them in Book II of his "Metrica", and centuries later, Pappus obtained a copy and wrote about them in the fifth book of his "Synagoge" or "Collection".

MH: You make no mention of them in any of your extant works; why is that?

A: Because it was practically my last investigation. War came to Syracuse in your year 215 B.C.E. and my job was to help with the defense of the city during the Roman siege. Also, Pappus wrote so well that after him, nobody bothered to copy my original.

One more thing. In your poster, Johnson solid J37, the elongated square gyrobicupola — a wonderful name! — is obtained from my rhombocuboctahedron by rotating the cap of four equilateral triangles and five squares one-eighth of a full turn. Three squares and one triangle meet at each corner, so technically, there should be *fourteen* Archimedean solids, not thirteen. I can fix that. By the way, archeologists have found cuboctahedral earrings in Germany dating from 450 C.E. Also, a centuries-old rhombocuboctahedral die was recently found in England, with six of its square faces marked with letters and the other twelve numbered in the manner of modern dice. They're everywhere, it seems.

MH: Big question: did you invent calculus?

Archie: Me? Certainly not. It is true that some of my methods foreshadow both calculus and analytic geometry. For example, in "On Spirals" I found the area swept out by the spiral $r = a\theta$ in one complete rotation to be one-third the area of the bounding circle. To do this, I sliced the bounding circle into an equal number of sectors ...

MH: ... rather like a pizza, wouldn't you say?

Archie: Exactly so. I approximated each segment of the spiral by a segment of a circle and added up those areas two different ways, one using the shortest ray inside the spiral segment and one using the longest such ray. This anticipates both the use of polar coordinates and using upper and lower sums to bound the integral of an increasing function. In "Quadrature of the Parabola" I show that the area of a parabolic segment is equal to four-thirds the area of a special triangle inscribed in that segment. The proof involves showing that a sum of areas of certain triangles nested within the parabolic segment is equal to

$$1+\frac{1}{4}+\frac{1}{4^2}+\cdots+\frac{1}{4^n},$$

which, I then show, is very close to $4/3$ for large n. Then I use Eudoxus' Method of Exhaustions to finish the proof. The Method of Exhaustions is what we used in those days because the idea of a limit had not occurred to us. All of the computations involving curvilinear surfaces and regions were my crude attempts at what eventually became the integral calculus.

MH: Which of your many achievements do you consider your best work?

A: Which of your children is your "best work"? I loved all of my works: On Floating Bodies, On Balancing Planes, Quadrature of the Parabola, On Spiral Lines, On Spheroids and Conoids, Measurement of a Circle, The Sand Reckoner, and the Cattle Problem. There are also the Stomachion and the Method of Mechanical Theorems, which you know only through the Palimpsest. There are the works you don't know about, but I'm not allowed to talk about them. Peculiar rules here, you know. But my favorite is The Sphere and Cylinder, in which I use the Method of Exhaustions to prove that the ratio of the volume of a sphere to the volume of its circumscribing cylinder is 2/3. That's why I gave instructions that it be inscribed on my gravestone. And Cicero found it many years later.

Now, I'm afraid my time here is short, Friend, and we haven't even talked about the Method, or the Stomachion, or several other works. You just might find me in your office again some time.

MH: I look forward to that, Sir. Finally, do you have any words of wisdom for our students?

A: Why, certainly. Read the Masters, never lose your sense of wonder, and enjoy the beauty of mathematics.

And he disappeared.

A moment later, I looked up at my wall and did a double-take. There were now *fourteen* Archimedean Solids on that poster, and the space formerly occupied by the Johnson Solid J37 was blank!

Well, Archimedes *did* say he could fix it, didn't he?

Suggested Reading

A wonderful place to start is Reviel Netz and William Noel's "The Archimedes Codex" (Da Capo Press, 2007). Their list of Further Reading will take you far. As for textbooks, Burton's "The History of Mathematics: An Introduction" (sixth edition:

McGraw-Hill, 2007) includes a good introduction to Archimedes' work. Finally, you can see the Polyhedra poster at `www.peda.com/poly`.