BSHM Bulletin, 2016
Vol. 31, No. 1, 1–14, http://dx.doi.org/10.1080/17498430.2015.1046037

Commutativity and collinearity: a historical case study of the interconnection of mathematical ideas. Part I

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This two-part paper investigates the discovery of an intriguing and fundamental connection between the famous but apparently unrelated mathematical work of two late third-century mathematicians, a link that went unnoticed for well over 1500 years. In this, the first installment of the paper, we examine the initial chain of mathematical events that would ultimately lead to the discovery of this remarkable link between two seemingly distinct areas of mathematics, encompassing contributions by a variety of mathematicians, from the most distinguished to the relatively unknown.

Introduction

The purpose of this paper is to shed light on a hitherto largely overlooked connection between the work of two well-known mathematicians of late antiquity, Diophantus (*c.* 250 CE) and Pappus (*c.* 320 CE). Judging from their surviving work, their mathematical interests appear to have been very different: Diophantus's *Arithmetica* contains only problems that would be described today as 'algebraic' or 'number theoretic', while Pappus's *Synagoge* is largely concerned with preserving and elucidating the ancient Greek geometrical tradition. Since neither the *Arithmetica* nor the *Synagoge* contain any reference to each other in any of the versions currently extant, there is no indication that either mathematician was acquainted with the other's work. Indeed, although almost nothing is known about the lives of these two men—except possibly that they either lived in or were in some way connected with the city of Alexandria—it is unlikely that they were even alive at the same time. Moreover, it would appear that neither Pappus nor Diophantus were particularly influential during what remained of antiquity, with full recognition coming only with the publication of Latin translations of their work during the sixteenth and seventeenth centuries.

Of all the mathematics contained in the *Synagoge*, the most famous result is buried deep in Book VII, being an amalgam of two consecutive propositions [Pappus 1986, 270–273]:

> Proposition 138: ... if AB and $\Gamma\Delta$ are parallel, and some straight lines $A\Delta$, AZ, $B\Gamma$, BZ intersect them, and $E\Delta$ and $E\Gamma$ are joined, it results that the [line] through H, M, and K is straight.
> Proposition 139: But now let AB and $\Gamma\Delta$ not be parallel, but let them intersect at N. That again the [line] through H, M, and K is straight.

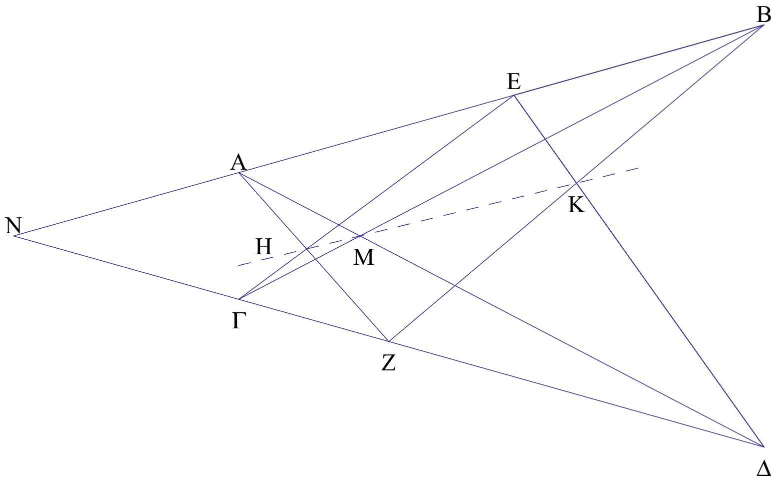

Figure 1. Pappus' theorem

In other words, given two straight lines in the plane with three arbitrary points on each, if lines are drawn from each point to two points on the opposite line, their intersections meet in three collinear points, as shown in Figure 1.

Now known as Pappus' Theorem, in the *Synagoge* this result was a mere isolated curiosity. Its true significance was only fully appreciated in the nineteenth century, when the rise of projective geometry revealed it to be a theorem of fundamental importance.[1]

In the *Arithmetica* there is a result that is even harder to find, since it is neither stated or proved as a theorem, but merely occurs as an implicit tool in the solution of a problem. Like the *Synagoge*, Diophantus' text was also a mathematical compendium, but its approach was completely different, having more in common stylistically with Chinese or Mesopotamian problem texts than with classical Greek geometry. It consisted originally of thirteen books, of which only six have survived, each containing a variety of problems, whose solutions involve numerical answers to determinate and indeterminate equations.[2] Many of these problems involved matters which would later become important in number theory, in particular those concerning sums of squares.

For example, in Book III, Problem 22, Diophantus asks the reader: 'To find four numbers such that the square of their sum plus or minus any one singly gives a square'[3] (Bachet 1621, 169). Midway through his solution, he notes that 'by its very nature 65 is divided into two [integer] squares in two ways, namely into 16 & 49 and 64 & 1, which is due to the fact that 65 is the product of 13 and 5, each of which is the sum of two squares'[4] (Bachet 1621, 169–170). In other words, because $65 = 13 \times 5 = (3^2 + 2^2)(2^2 + 1^2)$ it can also be written both as $(3 \cdot 2 - 2 \cdot 1)^2 + (2 \cdot 2 + 3 \cdot 1)^2$ *and* as $(3 \cdot 2 + 2 \cdot 1)^2 + (2 \cdot 2 - 3 \cdot 1)^2$. Although Diophantus gives only an illustrative

[1]For a good introduction to Pappus's Theorem and some of its ramifications, see Marchisotto 2002.
[2]For recent analysis of Diophantus's algebra and methods of solution, see Christianidis 2007, Bernard and Christianidis 2012, and Christianidis and Oaks 2013.
[3]'*Invenire quatuor numeros compositi ex omnibus quadratus, singulorum tam adiectione quam detractione faciat quadratum.*' Heath 1910, 166 gives this as Book III, Problem 19.
[4]'*Adhuc autem suapte natura numerus 65 dividitur bis in duos quadratos, nempe in 16 & 49 et rursus in 64 & 1, quod ei contingit quia fit ex multiplicatione mutua 5 & 13 quorum vterque in duos dividitur quadratos.*'

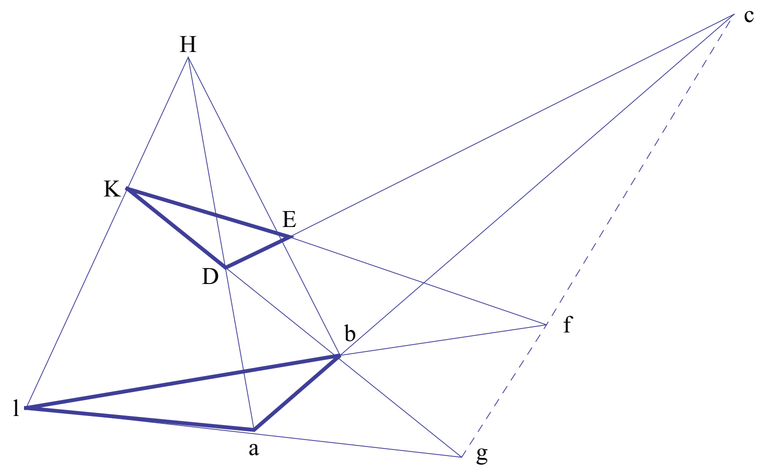

Figure 2. Desargues' theorem

example rather than a general result, implicit here is the understanding of a key number-theoretic fact, namely, that the product of the sums of two squares can be written as the sum of two squares, or in modern notation:

$$\left(a^2+b^2\right)\left(c^2+d^2\right)=(ac\mp bd)^2+(bc\pm ad)^2. \tag{1}$$

This identity appears to have been noticed explicitly in the mid-tenth century in a commentary on this particular Diophantine problem by al-Khazin (Rashed 1979, 213–217) and a proof of it was given by Fibonacci in his *Liber quadratorum* in 1225 (Sigler 1987, 23–31). But throughout the medieval period, the actual text of the *Arithmetica* remained unavailable to European scholars. In 1570, Raphael Bombelli made the first translation from Greek into Latin, but this was never published. And although the first printed edition of the *Arithmetica* appeared in 1575, the best known Latin translation was that of Claude Bachet (1581–1638) in 1621, which became for many years the canonical version. In his commentary to Book IV, Bachet remarked that Diophantus must have assumed that any number (meaning integer) is the sum

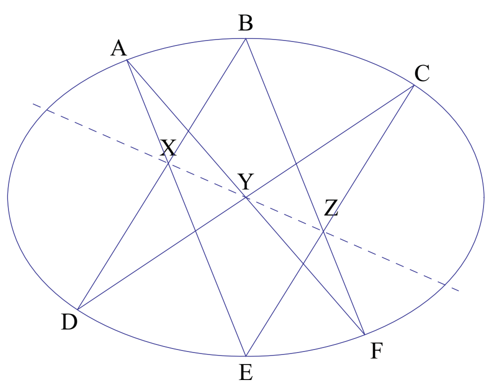

Figure 3. Pascal's theorem in an ellipse

of at most four squares, and noted that he had verified his hypothesis for all integers up to 325 (Bachet 1621, 241–242). This conjecture would not be proved for another 150 years, but Bachet's work on Diophantus was to serve as a catalyst for the growth of interest in number theory which would lead eventually to a link with Pappus' Theorem.

Meanwhile, Pappus' *Synagoge* had remained similarly unknown in western Europe until roughly the same time as the *Arithmetica*, finally receiving a Latin translation by Federico Commandino in 1588. The re-discovery of the material contained in Pappus contributed to a resurgence of interest in geometry in the seventeenth century. Along with this came two geometric contributions that, in common with those of Pappus and Diophantus over a millennium earlier, had little immediate effect, but were every bit as profound. The first was a pioneering work on projective geometry by the French engineer and architect Girard Desargues (1591–1661). Desargues' Theorem, discovered in the 1630s but not published until 1648, was widely overlooked until the mid-nineteenth century. It concerns two triangles related by projection from a point (see Figure 2): 'When the straight lines *HDa*, *HEb*, *cED*, *lga*, *lfb*, *HlK*, *DgK*, *EfK*, [and *abc*], lying in different planes or in the same one, meet in any order or angle in similar points; the points *c*, *f*, *g* lie on a straight line *cfg*'[5] (Desargues 1864, 413; Field and Gray 1987, 161).

One of the only seventeenth-century mathematicians to appreciate Desargues' work was the young Blaise Pascal (1623–62) who, aged only sixteen, derived a theorem, today called Pascal's Theorem, published in a small booklet entitled *Essai pour les coniques* in 1640, and similarly ignored.

> If a hexagon is inscribed in a conic, then the opposite sides intersect in three collinear points. (Smith 1959, 326)

This result is not just important as another foundational result in projective geometry, but because it is actually a generalization of Pappus' Theorem, which is the same result in the special case of a 'degenerate' conic section made up of two straight lines. Again, the full implications of this result would not become clear until many years had passed.

Thus in two very distinct time periods and geographical areas, we have observed mathematicians engaged in two correspondingly different areas of mathematics:

- Firstly, in third–fourth-century Alexandria, Diophantus worked on sums of squares, in which the algebraic identity (1) was used implicitly, while a major result of Pappus dealt with the intersection of lines and points in space; neither subject seems to have anything to do with the other.
- Secondly, in seventeenth-century France, on the one hand, the interests of Claude Bachet lay in arithmetical areas, including the Diophantus-inspired sums of squares, while on the other, the work of his contemporaries Desargues and Pascal was more geometrically-oriented, concerning collinearity and projections. Again, it is hard to see any formal connection between the number theory of the former and the geometry of the latter.

[5]'*Quand les droites HDa, HEb, cED, lga, lfb, HlK, DgK, EfK, soit en divers plans soit en un mesme, s'entrerencontrent par quelconque ordre ou biais que ce puisse estre, en de semblables points; les points c, f, g sont en une droite cfg*'.

Yet it turns out that there is indeed a very simple and profound relationship between the number theory of Diophantus and Bachet and the projective geometry of Pappus, Desargues and Pascal. But it would be many years before this connection was realized, and even today, it is far from common mathematical knowledge. This paper aims to provide a remedy.

To do this, we will trace the histories of number theory and projective geometry to find their eventual point of intersection. This will involve necessary historical detours into the respective developments of related mathematical areas, both obvious and unexpected. In this, Part I of our paper, we look at the development of number theory and the related subject of normed algebras. This will lead in Part II (Rice and Brown 2015) to a discussion of combinatorics and its connection to projective geometry, before we eventually arrive at the answer to our question: how are the apparently unrelated topics of Pappus' Theorem and Diophantus' implicit identity on the sums of squares ultimately linked?

Number theory: sums of squares

Although Diophantus' solution to Book III, Problem 22 showed him to have known implicitly that the product of the sums of two squares can be re-written as the sum of two squares, he was comparatively silent on matters concerning sums of three squares, probably because this is a much thornier issue. For example, as Albert Girard noticed in 1625 (Dickson 1919–23, vol 2, 276), integers such as 7, 15, 23, 28, 31 and 39 cannot even be written as sums of three squares. Also, whereas $3 = 1^2 + 1^2 + 1^2$ and $13 = 0^2 + 2^2 + 3^2$ both have three-square sum representations, their product 39 does not. Consequently, an extension of identity (1) to give a formula for products of sums of three squares is impossible.

This difficulty in constructing problems involving sums of three squares probably explains why they are essentially absent from the *Arithmetica*[6] and why Diophantus focused instead on problems of four squares. The particular Diophantine question that appears to have prompted Bachet's conjecture that any positive integer is the sum of at most four squares was Book IV, Problem 31 ('To find four square numbers such that their sum added to the sum of their sides makes a given number'), although other problems in the *Arithmetica* also seem to rely on this assumption, particularly Book IV, Problem 32 and Book V, Problem 17. In his text, Bachet provided representations of each positive integer up to 120 as sums of four squares or fewer, remarking that he would welcome a proper proof of his conjecture (Bachet 1621, 241–242).

The influence of Bachet's edition of Diophantus on Pierre de Fermat (1601–65) is well known, and much of the latter's work in number theory was stimulated by his reading of the text, with perhaps the classic example being Book II, Problem 8 ('To divide a given square number into two squares') which led him to formulate one of the most celebrated conjectures in the history of mathematics, the proof of which 'this margin is too small to contain'[7] (Diophantus 1670, 51). Fermat also appears to

[6]The one apparent exception to this is Book V, Problem 14 ('To divide unity into three parts such that, if we add the same number to each of the parts, the results are all squares'), which is equivalent to finding three square numbers, each bigger than a given number n, that sum to $3n + 1$.

[7]'*Hanc marginis exiguitas non caperet*'.

have tried to apply his method of infinite descent to proving Bachet's conjecture, but apparently without success (Dickson 1919 −23, vol 2, 276).

After Fermat, the next major steps to be taken with regard to Bachet's conjecture (and indeed in the subject of number theory itself) were by Leonhard Euler (1707−83). Euler's fascination with number theory appears to have been stimulated via his extensive correspondence with Christian Goldbach (1690−1764), who drew his attention to Fermat's vast array of unproved number-theoretic conjectures. In one of the earliest of these letters, dated 25 June 1730, Euler admitted that he was unable to prove Bachet's conjecture.[8] Nevertheless, over the course of the next few decades, he was to provide numerous related results that contributed to the eventual solution of the problem. One of these was discovered sometime between 1736 and 1740 (Matvievskaya 1960, 145; Pieper 1993, 14) and announced to Goldbach in a letter of 4 May 1748 (Fuss 1843, vol 1, 452). This was Euler's famous extension of formula (1), which gave the fundamental identity for products of sums of four squares:

$$\begin{cases} \left(a^2+b^2+c^2+d^2\right)\left(p^2+q^2+r^2+s^2\right)=A^2+B^2+C^2+D^2; \\ A=ap+bq+cr+ds, \qquad B=aq-bp\pm cs\mp dr, \\ C=ar\mp bs-cp\pm dq, \qquad D=as\pm br\mp cq-dp. \end{cases} \tag{2}$$

The paper in which this formula first appeared in print was written in 1751, although it was not published until 1760 (Euler 1760, 369).[9] In it, among other things, Euler effectively showed that quadratic residues of a prime p form a subgroup of index 2 of the multiplicative group of F_p. His ultimate aim in this paper was to prove Bachet's conjecture, but the best he could manage was to prove the result for rational, rather than integral, squares.

Nevertheless, Euler's work turned out to be a key to the eventual proof of Bachet's conjecture. In 1770, Joseph Louis Lagrange (1736−1813) gave the first full proof using a sophisticated argument, in which he explicitly acknowledged his reliance on many of the ideas in Euler's 1760 paper (Lagrange 1770, 190). Since every positive integer is a product of primes, Lagrange's proof hinged on showing that every prime number is a sum of four squares and then invoking formula (2). This appears to have prompted Euler to return to the subject and in 1773, using ideas he had developed as early as the 1740s (Matvievskaya and Ozhigova 1983, 157), he was able to provide a much simpler proof (Euler 1773), which, via its inclusion in Adrien-Marie Legendre's influential *Essai sur la théorie des nombres* (Legendre 1798, 198−202), became widely circulated.

Up to this point, all of the principal contributions to the subject had been made by mathematicians whose names are universally familiar to historians of mathematics. We now come to a development of equal magnitude made by a mathematician who, unlike his precursors, is relatively unknown. Carl Ferdinand Degen (1766−1825) was a scholar of exceptional and multifaceted abilities, excelling in theology, languages, the mathematical sciences, and philosophy, in which he received a doctorate in 1798. Although born in Germany, Degen spent the vast majority of his life in Denmark, being elected to the Royal Danish Academy of Sciences and Letters in 1800. Initially employed as a schoolteacher, he was instrumental in the creation of

[8] '*Theorema, quod quicunque numerus sit summa quatuor quadratorum, demonstrare non possum* ...' (Fuss 1843, vol 1, 30).

[9] Euler's four-squares formula also appeared in a later paper on orthogonal substitutions (Euler 1771, 311).

an expanded mathematics curriculum in Danish schools, and in 1814 he became professor of mathematics at the University of Copenhagen. His appointment raised the caliber of mathematics taught at the university and set the stage for the establishment of a mathematical research ethos in Denmark. By the end of the nineteenth century, his compatriot and fellow mathematician H G Zeuthen was able to remark: 'When looking at the development of mathematics in our country, starting with Degen, we dare say that his work has borne good fruit' (Zeuthen 1887–1905, 226).

Degen's achievements at the institutional level are reflected in the quality of mathematics he produced in his research publications, which included papers on calculus, mechanics, and geometry. These displayed not only a thorough familiarity with the work of Euler, Lagrange, and Legendre, but also a high level of independent research capability. One of his most highly regarded publications was a book entitled *Canon Pellianus* (Degen 1817), which tabulated solutions of the Pell equation $x^2 - ay^2 = 1$ for all non-square positive integers a up to and including 1000, as well as those solutions of $x^2 - ay^2 = -1$ (when solvable) for $1 \leq a \leq 1000$. This book had almost certainly been inspired by his reading of Euler, who had devoted much research to the Pell equation—even to the extent of endowing it with its erroneous name (see Dickson 1919–23, vol 2, 354).

Being also intimately connected to sums of squares, it is highly likely that Degen's work on the Pell equation resulted in a paper he wrote in 1818 and presented to the St Petersburg Academy of Sciences, where Euler had published so much of his research and to which Degen would be elected a corresponding member in 1819. This publication venue was certainly appropriate as Degen's paper contained a highly significant addition to Eulerian number theory. After beginning with a recapitulation of Euler and Lagrange's recent results on sums of four squares, Degen's paper then announced nonchalantly (Degen 1822, 209) 'the extension of Euler's theorem to sums of eight squares', giving the following formula:[10]

$$\left\{\begin{array}{l} (P^2+Q^2+R^2+S^2+T^2+U^2+V^2+X^2) \\ \quad\times\left(p^2+q^2+r^2+s^2+t^2+u^2+v^2+x^2\right) \\ =(Pp+Qq+Rr+Ss+Tt+Uu+Vv+Xx)^2 \\ +(Pq-Qp+Rs-Sr+Tu-Ut+Vx-Xv)^2 \\ +(Pr-Qs-Rp+Sq\mp Tv\pm Ux\pm Vt\mp Xu)^2 \\ +(Ps+Qr-Rq-Sp\pm Tx\pm Uv\mp Vu\mp Xt)^2 \\ +(Pt-Qu\pm Rv\mp Sx-Tp+Uq\mp Vr\pm Xs)^2 \\ +(Pu+Qt\mp Rx\mp Sv-Tq-Up\pm Vs\pm Xr)^2 \\ +(Pv-Qx\mp Rt\pm Su\pm Tr\mp Us-Vp+Xq)^2 \\ +(Px+Qv\pm Ru\pm St\mp Ts\mp Ur-Vq-Xp)^2 \end{array}\right. \tag{3}$$

As to how he came upon this marvellous result, Degen's paper gives us no clue. The prose is dry and terse, and its author provides the reader with no great insights regarding his mode of discovery. It is probable, as van der Blij (1961, 107) guesses, that Degen found the eight-squares identity simply by trial and error. Whatever the

[10]Degen's formula contains a misprint in which the sign of Rt is given as $\pm$. This error was first pointed out by Dickson 1919, 164.

method, a pattern was now apparent: since formulae clearly existed for products of sums of 1, 2, 4 and 8 squares, Degen was convinced that similar identities must exist for sums of any $n = 2^m$ squares. Having disposed of $n = 8$, he immediately threw himself into the case of 16 squares, only to be defeated by the laborious and tedious process of determining the parity of all 256 terms. Subsequent mathematicians would also investigate the possibility of a formula for 2^m squares during the following decades, but none were successful for the simple reason that no such formula exists for $m > 3$. But the discovery of this fact would have to wait until the very end of the nineteenth century when it appeared in a different, but related, context.[11]

Normed algebras: the search for hypercomplex numbers

Despite the significance of the result, Degen's discovery of the eight-squares identity seems to have gone largely unnoticed at the time and, as L E Dickson pointed out in 1919, 'this paper has been overlooked by all subsequent writers on the subject' (Dickson 1919, 164). Nevertheless, Degen's formula (3) would re-appear a quarter of a century after its original discovery when it was independently re-discovered *twice*: by John Graves (1806–71), an Irish mathematically-trained lawyer working in London, and by Arthur Cayley (1821–95), an English mathematician soon to become a lawyer, working in Cambridge.

The context in which their discoveries were made was the search for higher-dimensional systems of complex numbers. To explain what they were looking for, we use the following (modern) definition:

A *normed algebra* $\mathbb{A}$ is an n-dimensional vector space over the real numbers $\mathbb{R}$ such that

- $\alpha(xy) = (\alpha x)y = x(\alpha y)$, for all $\alpha \in \mathbb{R}, x, y \in \mathbb{A}$
- $x(y + z) = xy + xz$, $(y + z)x = yx + zx$, for all $x, y, z \in \mathbb{A}$
- there exists a function $N : \mathbb{A} \to \mathbb{R}$ such that $N(xy) = N(x)N(y)$, for all $x, y \in \mathbb{A}$.

If we define the *norm* of $x \in \mathbb{R}$ to be $N(x) = x^2$, then one sees that $\mathbb{R}$ is a one-dimensional normed algebra. Furthermore, defining the *norm* of $z = a + bi \in \mathbb{C}$ to be $N(z) = a^2 + b^2$, then given any other $w = c + di \in \mathbb{C}$, with $N(w) = c^2 + d^2$ and where $zw = (ac - bd) + (ad + bc)i$ as usual, we see not only that $\mathbb{C}$ is a two-dimensional normed algebra, but also that

$$N(zw) = (ac - bd)^2 + (ad + bc)^2 = (a^2 + b^2)(c^2 + d^2) = N(z)N(w),$$

which is essentially formula (1). In other words, the two-squares identity implicitly contains the rule for the multiplication of two complex numbers.

Despite the long and reluctant process by which mathematicians slowly came to accept the legitimacy of complex numbers, by the early nineteenth century an intuitive feeling had emerged that new hierarchies of 'hypercomplex' numbers might be possible. For example, in a paper of 1813, Jean-Robert Argand (1768–1822), had erroneously argued that i^i could not be written in the form $a + bi$, and actually belonged to a new system of hypercomplex numbers requiring a third dimension (Argand 1813).[12]

[11] A good survey article on the history of n-square identities is Hollings 2006.

[12] In doing so, he was presumably ignorant of Euler's proof that $i^i = e^{-(\pi/2) \pm 2n\pi}$ (Euler 1751, 130–133).

The story of the discovery in 1843 of the first actual hypercomplex number system, the quaternions, by William Rowan Hamilton (1805–65) is very well known,[13] but for the purposes of this paper a few aspects are worth mentioning. After spending much of the 1830s in search of a three-dimensional representation of the complex number triple

$$z = a + bi + cj,\ \ a, b, c \in \mathbb{R},\ \ i^2 = j^2 = -1,\ \ i \neq j,$$

Hamilton was unable to create an algebra that was closed under multiplication. It was not until he let $ij = -ji$ and defined this product to be 'some new sort of unit operator' (Hamilton 1853, 144) k, such that $jk = -kj = i$ and $ki = -ik = j$, that he realized, firstly that $k^2 = ijk = (ij)(ij) = (ij)(-ji) = -i(j^2)i = -i(-1)i = -(-1)(i^2) = -1$, secondly that his new algebra, $\mathbb{H}$, was four-dimensional, consisting of *quaternions* of the form

$$z = a + bi + cj + dk,\ \ a, b, c, d \in \mathbb{R},\ \ i^2 = j^2 = k^2 = ijk = -1,$$

and thirdly, that $\mathbb{H}$ did not obey the commutative law of multiplication.

Hamilton's fundamental formula $i^2 = j^2 = k^2 = ijk = -1$ led to the following rule for multiplying two quaternions together (Hamilton 1843, 108):

$$\begin{aligned}&(a + bi + cj + dk)(\alpha + \beta i + \gamma j + \delta k)\\ &= (a\alpha - b\beta - c\gamma - d\delta) + (a\beta + b\alpha + c\delta - d\gamma)i\\ &\quad + (a\gamma - b\delta + c\alpha + d\beta)j + (a\delta + b\gamma - c\beta + d\alpha)k,\end{aligned}$$

and, by extension of the definition of the norm to quaternions, i.e. $N(z) = a^2 + b^2 + c^2 + d^2$, it followed (Hamilton 1843, 109) that

$$\begin{aligned}&\left(a^2 + b^2 + c^2 + d^2\right)\left(\alpha^2 + \beta^2 + \gamma^2 + \delta^2\right)\\ &= (a\alpha - b\beta - c\gamma - d\delta)^2 + (a\beta + b\alpha + c\delta - d\gamma)^2\\ &\quad + (a\gamma - b\delta + c\alpha + d\beta)^2 + (a\delta + b\gamma - c\beta + d\alpha)^2.\end{aligned}$$

In other words, not only had Hamilton proved that the quaternions $\mathbb{H}$ are a normed algebra, but he had also re-discovered Euler's formula (2) in the process.

At this stage, neither Hamilton nor his friend John Graves, who had been similarly searching for three-dimensional algebraic triples during the 1830s and early 1840s, were aware of Euler's four-squares formula or of the impossibility of a three-square identity, which ironically had rendered their initial quest impossible. But on 22 January 1844, three months after Hamilton's discovery in October 1843, Graves (1844) reported that

> On Friday last I looked into Lagrange's Théorie des Nombres[14] and found for the first time that I had lately been on the track of former mathematicians. For

[13] Those unfamiliar with it may wish to consult the accounts given in Crowe 1985, Hankins 1980, and van der Waerden 1976.

[14] Graves mistakenly wrote Lagrange, although he meant Legendre 1798.

example, the mode by which I satisfied myself that a general theorem

$$\left(x_1{}^2+x_2{}^2+x_3{}^2\right)\left(y_1{}^2+y_2{}^2+y_3{}^2\right)=z_1{}^2+z_2{}^2+z_3{}^2$$

was impossible was the very mode mentioned by Legendre, who gives the very example that occurred to me, viz., $3\times 21=63$, it being impossible to compound 63 of three squares. I then learned that the theorem

$$\left(x_1{}^2+x_2{}^2+x_3{}^2+x_4{}^2\right)\left(y_1{}^2+y_2{}^2+y_3{}^2+y_4{}^2\right)=z_1{}^2+z_2{}^2+z_3{}^2+z_4{}^2$$

was Euler's...[15]

This did not deter other mathematicians (including, for example, Augustus De Morgan and Graves' brother Charles) from continuing to search for the elusive 'triple algebras', and producing some interesting, if bizarre, results along the way (for example, De Morgan 1844; Graves 1847). And although John Graves was initially skeptical 'as to the extent to which we are at liberty arbitrarily to create imaginaries, and to endow them with supernatural properties' (Graves 1882–89, vol 2, 443), Hamilton's discovery/invention of quaternions clearly prompted him to do just that. By Christmas 1843, he had succeeded in taking Hamilton's bold abstraction four dimensions further. On 26 December 1843, in a letter to his friend, Graves announced the discovery of a new system of algebra, $\mathbb{O}$, which he called 'octaves', consisting of the eight base units 1, i_1, i_2, i_3, i_4, i_5, i_6, i_7 such that[16]

$$\begin{aligned}
&i_1{}^2=i_2{}^2=i_3{}^2=i_4{}^2=i_5{}^2=i_6{}^2=i_7{}^2=-1\\
&i_1=i_2i_3=i_4i_5=i_7i_6=-i_3i_2=-i_5i_4=-i_6i_7\\
&i_2=i_3i_1=i_4i_6=i_5i_7=-i_1i_3=-i_6i_4=-i_7i_5\\
&i_3=i_1i_2=i_4i_7=i_6i_5=-i_2i_1=-i_7i_4=-i_5i_6\\
&i_4=i_5i_1=i_6i_2=i_7i_3=-i_1i_5=-i_2i_6=-i_3i_7\\
&i_5=i_1i_4=i_7i_2=i_3i_6=-i_4i_1=-i_2i_7=-i_6i_3\\
&i_6=i_2i_4=i_1i_7=i_5i_3=-i_4i_2=-i_7i_1=-i_3i_5\\
&i_7=i_6i_1=i_2i_5=i_3i_4=-i_1i_6=-i_5i_2=-i_4i_3
\end{aligned}$$

Just as Hamilton's $i^2=j^2=k^2=ijk=-1$ formula enabled the closed multiplication of quaternions, the above rules did the same for Graves' octaves. Moreover, given the product of two octaves $z=a_0+a_1i_1+a_2i_2+a_3i_3+a_4i_4+a_5i_5+a_6i_6+a_7i_7$ and $w=b_0+b_1i_1+b_2i_2+b_3i_3+b_4i_4+b_5i_5+b_6i_6+b_7i_7$, with the norm defined in the usual way (that is, $N(z)=a_0{}^2+a_1{}^2+a_2{}^2+a_3{}^2+a_4{}^2+a_5{}^2+a_6{}^2+a_7{}^2$),

[15]Euler's formula (2) appears in Legendre 1798, 200, where Legendre remarks in a footnote: 'On peut s'assurer qu'il n'existe aucune formule semblable pour trois quarrés, c'est-à-dire que le produit d'une somme de trois quarrés par une some de trois quarrés, ne peut pas être exprimé généralement par une somme de trois quarrés. Car si cela étoit possible, le produit (1 + 1 + 1)(16 + 4 + 1), qui est 63, pourroit se décomposer en trois quarrés'.

[16]Graves' notation was actually 1, *i*, *j*, *k*, *l*, *m*, *n*, *o*. We use the notation employed in Cayley 1845.

Graves noticed that

$$
\begin{aligned}
&\left(a_0{}^2 + a_1{}^2 + a_2{}^2 + a_3{}^2 + a_4{}^2 + a_5{}^2 + a_6{}^2 + a_7{}^2\right) \\
&\times\left(b_0{}^2 + b_1{}^2 + b_2{}^2 + b_3{}^2 + b_4{}^2 + b_5{}^2 + b_6{}^2 + b_7{}^2\right) \\
&\quad = (a_0b_0 - a_1b_1 - a_2b_2 - a_3b_3 - a_4b_4 - a_5b_5 - a_6b_6 - a_7b_7)^2 \\
&\quad + (a_0b_1 + a_1b_0 + a_2b_3 - a_3b_2 + a_4b_5 - a_5b_4 - a_6b_7 + a_7b_6)^2 \\
&\quad + (a_0b_2 - a_1b_3 + a_2b_0 + a_3b_1 + a_4b_6 + a_5b_7 - a_6b_4 - a_7b_5)^2 \\
&\quad + (a_0b_3 + a_1b_2 - a_2b_1 + a_3b_0 + a_4b_7 - a_5b_6 + a_6b_5 - a_7b_4)^2 \\
&\quad + (a_0b_4 - a_1b_5 - a_2b_6 - a_3b_7 + a_4b_0 + a_5b_1 + a_6b_2 + a_7b_3)^2 \\
&\quad + (a_0b_5 + a_1b_4 - a_2b_7 + a_3b_6 - a_4b_1 + a_5b_0 - a_6b_3 + a_7b_2)^2 \\
&\quad + (a_0b_6 + a_1b_7 + a_2b_4 - a_3b_5 - a_4b_2 + a_5b_3 + a_6b_0 - a_7b_1)^2 \\
&\quad + (a_0b_7 - a_1b_6 - a_2b_5 + a_3b_4 - a_4b_3 - a_5b_2 + a_6b_1 + a_7b_0)^2
\end{aligned}
$$

that is, $N(z)N(w) = N(zw)$. Thus Graves had found another normed alegbra and, in so doing, re-discovered Degen's eight-square identity (3).[17]

As was clear from Graves' rules defining his base units, in common with quaternions, multiplication in $\mathbb{O}$ was noncommutative. But as Hamilton's nineteenth-century biographer[18] put it, by the summer of 1844, 'Hamilton had to report to his friend that the four-legged animal could stand better on his feet, and move in all directions better, than his later-born brother with eight legs' (Graves 1882–89, vol 2, 456). It appeared that, in order to extend the dimensions of his algebra, Graves had had to sacrifice another basic law of arithmetic: '*In general in my system of Quaternions* (containing only three imaginaries), it is *indifferent where we place the points, in any successive multiplication:* A.BC = AB.C = ABC, if A, B, C be quaternions: but not so, generally, with your Octaves'. In other words, since, for example, $(i_3i_4)i_5 = i_7i_5 = -i_2$ but $i_3(i_4i_5) = i_3i_1 = i_2$, octave multiplication was also nonassociative.

Graves was not the only mathematician to have been stimulated by Hamilton's creation of quaternions. In one of many similar occurrences throughout the history of mathematics where the same or equivalent discovery is made by two people with no common point of contact, just months after Graves' initial discovery, the octaves $\mathbb{O}$, their noncommutativity and nonassociativity, and the eight-squares formula (3) were all found independently by the young Arthur Cayley, then a recent graduate of the University of Cambridge. Sadly for Graves, since Cayley was the first to publish his findings (Cayley 1845, 1847) and subsequently went on to become a far more famous mathematician, Graves failed to receive as much recognition as perhaps was his due. Indeed, for many years, Graves' octaves were often referred to as 'Cayley numbers', although today, the name *octonions* is generally used.[19]

[17]See Addendum to Young 1848 in *Transactions of the Royal Irish Academy*, 21 (Part II) (1848), 338–341.
[18]John Graves' younger brother Robert.
[19]See, for example, Baez 2002, and Conway and Smith 2003. It must also have been disappointing for Graves when he eventually discovered that even the eight-squares formula had first been published by someone else. As he wrote to Hamilton on 4 December 1852: 'The theorem of eight squares, which I communicated to you some years ago, had, I find, been previously discovered by C. F. Degen, "*Adumbratio Demonstrationis Theorematis Arithmetici maxime generalis.*" *Mémoires de l'Académie Impériale des Sciences de St. Petersburg*, tom. viii. p. 207, St. Petersburg, 1822. Conventui exhibuit die 7 Oct. 1818.'—Graves 1882–89, vol 2, 577n

After this, a variety of new hypercomplex number systems began to emerge. In 1844, Hamilton created the algebra of *biquaternions*,[20] similar to quaternions but of the form

$$z = a + bi + cj + dk, \ \ a, b, c, d \in \mathbb{C},$$

but this was not a normed algebra (Hamilton 1853, 650). That year also saw the first publication of Hermann Grassmann's (later-to-be) influential *Ausdehnungslehre*, which introduced notions now fundamental to linear algebra and the geometry of n-dimensional space. It was under the influence of Grassmann's ideas that, in 1878, William Clifford gave a generalization for hypercomplex number systems with 2^n base components $1, i_1, i_2, \cdots i_{2^n-1}$ (where ${i_\alpha}^2 = -1$ and $i_\alpha i_\beta = -i_\beta i_\alpha$ for $\alpha \neq \beta$) now known as 'Clifford algebras' (Clifford 1878). Georg Frobenius (1878) quickly proved that all such algebras with $n > 2$ would be nonassociative, but were there any other *normed* algebras over $\mathbb{R}$ for $n > 3$? As early as the 1840s, mathematicians (for example, Kirkman 1848; Young 1848; Cayley 1852) had suspected that the answer to this question was no.

The situation, then, was exactly identical to that with regard to identities for products of sums of n squares, with which, by now, mathematicians had made the straightforward connection. Little did they realize, however, that a new branch of mathematics, then just at its inception, would also have a role to play and that this would ultimately lead to a hitherto unrealized connection with Pappus' Theorem. This connection and the story of its discovery will be found in the concluding part of this paper.

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[20]It should be noted that Hamilton's biquaternions are very different from the algebra of the same name introduced by Clifford in 1873, which are of the form $p + \omega q$, where p and q are real quaternions, ω commutes with every real quaternion, and $\omega^2 = 0$ or $\omega^2 = 1$.

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