ABSTRACT

This dissertation studies connections between the preprojective representations of a finite connected valued quiver without oriented cycles, the (+)-admissible sequences of vertices, and the Weyl group. For each preprojective representation, a shortest (+)-admissible sequence annihilating the representation is unique up to a certain equivalence. A (+)-admissible sequence is the shortest sequence annihilating some preprojective representation if and only if the product of simple reflections associated to the vertices of the sequence is a reduced word in the Weyl group. These statements have the following application that strengthens known results of Howlett and Fomin-Zelevinsky. For any fixed Coxeter element of the Weyl group associated to an indecomposable symmetrizable generalized Cartan matrix, the group is infinite if and only if the powers of the element are reduced words. These results also extend Gabriel's Theorem by providing a one-to-one correspondence between indecomposable preprojective representations and elements in the Weyl group that have a reduced expression whose associated sequence of vertices is a principal (+)-admissible sequence.

Representations of a Valued Quiver, the Lattice of Admissible Sequences, and the Weyl Group of a Kac-Moody Algebra

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The packet you'll get from the graduate school will tell you about the committee approval page, which is what everyone signs. When the time comes, you'll replace this page with that one.

Contents

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Chapter 1

Introduction

In [Gab72], Gabriel shows that a quiver is of finite representation type (meaning that it has finitely many non-isomorphic indecomposable representations) if and only if the underlying graph is a Dynkin diagram of type A_n, D_n, E_6, E_7, or E_8. When the quiver is of finite type, the representations are in one-to-one correspondence to the positive roots of the associated root system.

Bernšteĭn, Gel′fand, and Ponomarev [BGP73] give an alternate proof of the main result in [Gab72], known as Gabriel's Theorem. Their proof is closer in spirit to the root system, and they introduce a reflection functor that acts on the category of representations in a manner similar to the way the simple reflections in the Weyl group act on the positive roots of the root system. This reflection functor motivates the notion of (+)-admissible sequences. They also define two types of representations, preprojective and regular. Preprojective representations are those that are annihilated by some composition of reflection functors. Their proof of Gabriel's Theorem uses properties of preprojective representations.

Dlab and Ringel [DR76] extend the techniques developed in [BGP73] and generalize Gabriel's Theorem to valued quivers. They show that the valued quivers of finite type are exactly those whose underlying graph is a Dynkin diagram, i.e. a graph of type A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4, or G_2, and obtain a one-to-one correspondence between indecomposable preprojective representations and positive roots similar to that in Gabriel's Theorem.

Bernšteĭn, Gel′fand, and Ponomarev give two equivalent definitions of preprojective representations. Both [BGP73] and [DR76] focus on the first definition, while the work of Kleiner and Tyler in [KT05] and [KT06] focuses on the second definition. They use the techniques developed in [BGP73] to associate to each indecomposable preprojective representation a canonical (+)-admissible sequence, and show that the combinatorial structure of the set of (+)-admissible sequences carries a lot of information about the preprojective component of the Auslander-Reiten quiver.

This dissertation extends and improves the above results in the following manner. The poset of (+)-admissible sequences defined in [KT05] is given a (distributive) lattice structure. The join of sequences corresponds to the direct sum of the associated representations. This lattice structure enables us to extend the study of [KT05, KT06] to decomposable preprojective representations, and use properties of the Auslander-Reiten quiver to show that for any preprojective representation, the associated canonical (+)-admissible sequence gives a reduced word in the Weyl group. The connection between preprojective representations and reduced words in the Weyl group is developed to show that indecomposable preprojective representations are in one-to-one correspondence with elements in the Weyl group that have a reduced expression obtained from a principal (+)-admissible sequence. This result extends Gabriel's Theorem and the considerations of [DR76] to indecomposable preprojective representations of all valued quivers, regardless of the representation type.

This correspondence also allows us to use properties of preprojective representations to obtain information about the Weyl group. Zelevinsky brought to our attention the following two results. Howlett proved that a Coxeter group is infinite if and only if a Coxeter element has infinite order [How82, Theorem 4.1]. Fomin and Zelevinsky proved the following. Let the vertex set of a valued graph be bipartite, i.e., the set $\{1, \dots, n\}$ is a disjoint union of nonempty subsets I, J and, for $h \neq l$, no edge connects vertex h to vertex l if either $h, l \in I$ or $h, l \in J$. Letting $\{\sigma_i\}_{i=1}^n$ be the generators of the Weyl and setting $c = \prod_{i \in I} \sigma_i \prod_{j \in J} \sigma_j$, then the Weyl group is infinite if and only if the powers of c are reduced words [FZ06, Corollary 9.6]. Inspired by the latter, we prove Theorem 5.9, that a Coxeter group generated by a valued graph is infinite if and only if all powers of a Coxeter element are reduced words, which strengthens the aforementioned result of Howlett and generalizes the aforementioned result of Fomin-Zelevinsky.

The contents of each chapter are now described. Chapter 2 contains definitions of valued quivers and their representations. It also introduces (+)-admissible sequences and gives the two equivalent definitions of preprojective representations found in [BGP73]. Chapter 2 ends with a section showing how representations of quivers form a special case of the representations of valued quivers studied in [DR76].

Chapter 3 begins by reviewing some of the considerations of (+)-admissible sequences as developed in [KT05]. Specifically, Section 3.1 describes the equivalence of sequences as well as the canonical form of a sequence and relates these to the notion of the multiplicity of a vertex in a sequence, which we introduce. Section 3.2 recalls the partial order on equivalence classes of sequences, and the last section of Chapter 3 introduces a (distributive) lattice structure on the partially ordered set of (+)-admissible sequences. The connection between the lattice structure and the partial order is explored in Theorem 3.7.

Chapter 4 consists of four sections. The first section uses principal (+)-admissible sequences and the lattice structure of Chapter 3 to construct all (+)-admissible sequences. Section 4.2 reviews Kleiner and Tyler's method of associating a canonical (+)-admissible sequence to an indecomposable preprojective representation and extends this method to decomposable preprojective representations. Chapter 4 also recalls the definition of the translation quiver of the opposite quiver and the Auslander-Reiten quiver. Two results crucial to Chapter 5 are obtained. The first is that for two indecomposable preprojective representations, the partial order in the Auslander-Reiten quiver is equivalent to the partial order of their associated canonical principal

(+)-admissible sequences. The second crucial result deals with the form of principal (+)-admissible sequences.

Chapter 5 contains the main results of this dissertation. It begins by recalling the relation described in [BGP73] between the reflection functor and the simple reflections in the Weyl group. Properties of the Auslander-Reiten quiver are then used to prove that the canonical principal (+)-admissible sequence associated to a preprojective representation yields a reduced word in the Weyl group. The rest of Section 5.2 focuses on the interplay between indecomposable preprojective representations and words in the Weyl group. Theorem 5.7 of Section 5.3 is the main result of this dissertation, which establishes that for a (+)-admissible sequence, being associated to a preprojective representation is equivalent to yielding a reduced word in the Weyl group.

The last chapter of this dissertation contains material relating preprojective modules with other areas of mathematics. The first section of Chapter 6 contains ties to Coxeter-sortable elements of the Weyl group as well as non-crossing partitions and clusters. Section 6.2 mentions the weak Bruhat order defined on a Coxeter group and shows that, while this order is similar to that on (+)-admissible sequences, it not comparable to the order on (+)-admissible sequences. The last section quotes a result of Speyer in [Spe06]. After Speyer read our proof of Theorem 5.9 and the related material, he emulated our proof, removing the dependence on representation theory. In doing so, he proves a result that generalizes the aforementioned theorem of Howlett, obtaining the fact that all powers of a Coxeter element are reduced, as in Theorem 5.9, but in the slightly broader setting of Coxeter groups, as in Howlett's result.

Chapter 2

Preliminaries

2.1 Valued Quivers

We recall some facts, definitions, and notation, using freely [ARS97, BGP73, DR76, KT06]. A *graph* is a pair $\Gamma = (\Gamma_0, \Gamma_1)$, where Γ_0 is the set of vertices, and the set of edges Γ_1 consists of some two-element subsets of Γ_0. A *valuation* $\mathbf{b}$ of Γ is a set of integers $b_{ij} \geq 0$ for all pairs $i, j \in \Gamma_0$ where $b_{ii} = 0$; if $i \neq j$ then $b_{ij} \neq 0$ if and only if $\{i, j\} \in \Gamma_1$; and there exist integers $d_i > 0$ satisfying $d_i b_{ij} = d_j b_{ji}$ for all $i, j \in \Gamma_0$. The pair $(\Gamma, \mathbf{b})$ is a *valued graph*, which is called *connected* if Γ is connected.

An *orientation*, Λ, on Γ consists of two functions $s : \Gamma_1 \to \Gamma_0$ and $e : \Gamma_1 \to \Gamma_0$. If $a \in \Gamma_1$, then $s(a)$ and $e(a)$ are the vertices incident with a, called the starting point and the endpoint of a, respectively. The triple $(\Gamma, \mathbf{b}, \Lambda)$ is a *valued quiver*, and a is then called an *arrow* of the quiver. Given a sequence of arrows $a_1, \ldots, a_t,\ t > 0$, satisfying $e(a_i) = s(a_{i+1}),\ 0 < i < t$, one forms a *path* $p = a_t \dots a_1$ of length t in $(\Gamma, \mathbf{b}, \Lambda)$ with $s(p) = s(a_1)$ and $e(p) = e(a_t)$. By definition, for all $x \in \Gamma_0$ there is a unique path e_x of length 0 with $s(e_x) = e(e_x) = x$. We say that p is a path from $s(p)$ to $e(p)$ and write $p : s(p) \to e(p)$. A path p of length > 0 is an *oriented cycle* if $s(p) = e(p)$. The set of vertices of any valued quiver without oriented cycles (no finiteness assumptions) becomes a partially ordered set (poset) if one puts $x \le y$ whenever there is a path from x to y. If $(\Gamma, \mathbf{b}, \Lambda)$ has no oriented cycles, we denote this poset by (Γ_0, Λ). Throughout this dissertation, all orientations Λ, Θ, etc., are such that $(\Gamma, \mathbf{b}, \Lambda)$, $(\Gamma, \mathbf{b}, \Theta)$, etc., have no oriented cycles.

Example 2.1. When writing a valued quiver, the symbol $i \xrightarrow{(b_{ij}, b_{ji})} j$ will denote arrows and the corresponding integers of the valuation. If $b_{ij} = b_{ji} = 1$, the integers of the valuation will be omitted. In this notation, the following is an example of a valued quiver.

$$(\Gamma, \mathbf{b}, \Lambda) = \begin{array}{ccccc} 1 & \xrightarrow{(2,1)} & 2 & \xrightarrow{(1,3)} & 3 \\ & & \uparrow & & \\ & & 4 & & \end{array}$$

The valuation $\mathbf{b}$ may be written as a matrix $\mathbf{b} = (b_{ij}) = \left(\begin{smallmatrix} 0 & 2 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 3 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{smallmatrix}\right)$. Likewise, the valuation requires the existence of d_i's, which, when written in matrix form may be taken as $D = \left(\begin{smallmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{smallmatrix}\right)$. Notice that $\mathbf{b}$ is symmetrizable in the sense that $D\mathbf{b} = \left(\begin{smallmatrix} 0 & 2 & 0 & 0 \\ 2 & 0 & 3 & 1 \\ 0 & 3 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{smallmatrix}\right)$ is symmetric.

To define representations of a valued quiver $(\Gamma, \mathbf{b}, \Lambda)$, one has to choose a *modu-*

lation $\mathfrak{B}$ of the valued graph $(\Gamma, \mathbf{b})$, which by definition is a set of division rings $\mathbf{k}_i$, $i \in \Gamma_0$, together with a $\mathbf{k}_i - \mathbf{k}_j$-bimodule ${}_iB_j$ and a $\mathbf{k}_j - \mathbf{k}_i$-bimodule ${}_jB_i$ for each $\{i, j\} \in \Gamma_1$ such that

(i) there are $\mathbf{k}_j - \mathbf{k}_i$-bimodule isomorphisms

$$ {}_jB_i \cong \mathrm{Hom}_{\mathbf{k}_i}({}_iB_j, \mathbf{k}_i) \cong \mathrm{Hom}_{\mathbf{k}_j}({}_iB_j, \mathbf{k}_j) $$

and

(ii) $\dim_{\mathbf{k}_i}({}_iB_j) = b_{ij}$.

Unless indicated otherwise, for the rest of this dissertation we denote by Γ an arbitrary finite connected valued graph with $|\Gamma_0| > 1$, where $|X|$ stands for the cardinality of a set X, and with a valuation $\mathbf{b}$ and modulation $\mathfrak{B}$; denote by (Γ, Λ) the corresponding valued quiver with orientation Λ; and assume that $\dim_k \mathbf{k}_i < \infty$ for all i, where k is a common central subfield of the $\mathbf{k}_i$'s acting centrally on all bimodules ${}_iB_j$. Under the assumption, each ${}_iB_j$ is a finite dimensional k-space.

A (left) representation (V, f) of (Γ, Λ) is a set of finite dimensional left $\mathbf{k}_i$-spaces V_i, $i \in \Gamma_0$, together with $\mathbf{k}_j$-linear maps $f_a : {}_jB_i \otimes_{\mathbf{k}_i} V_i \to V_j$ for all arrows $a : i \to j$. The representation with $V_i = 0$ for all $i \in \Gamma_0$ is the *zero representation*, written $(V, f) = 0$. These representations become objects in the category $\mathrm{Rep}(\Gamma, \Lambda)$ of representations of the valued quiver (Γ, Λ). Morphisms between two objects in $\mathrm{Rep}(\Gamma, \Lambda)$ are maps $\phi : (V, f) \to (W, g)$ where ϕ is a collection of linear transformations $\phi_i : V_i \to W_i$ for each $i \in \Gamma_0$ such that for all arrows $a : i \to j$, the diagram

$$\begin{array}{ccc} {}_jB_i \otimes_{\mathbf{k}_i} V_i & \xrightarrow{f_a} & V_j \\ \downarrow {\scriptstyle 1\otimes_{\mathbf{k}_i}\phi_i} & & \downarrow {\scriptstyle \phi_j} \\ {}_jB_i \otimes_{\mathbf{k}_i} W_i & \xrightarrow[g_a]{} & W_j \end{array}$$

commutes, giving $\phi_j f_a = g_a(1 \otimes \phi_i)$.

Putting $\mathbf{k} = \prod_{i\in\Gamma_0} \mathbf{k}_i$ and viewing $B = \bigoplus_{i\to j} {}_jB_i$ as a $\mathbf{k}$-$\mathbf{k}$-bimodule where $\mathbf{k}$ acts on ${}_jB_i$ from the left via the projection $\mathbf{k} \to \mathbf{k}_j$ and from the right via the projection $\mathbf{k} \to \mathbf{k}_i$, one forms the tensor ring $\mathrm{T}(\mathbf{k}, B) = \bigoplus_{n=0}^{\infty} B^{(n)}$ where $B^{(n)} = B \otimes_{\mathbf{k}} \cdots \otimes_{\mathbf{k}} B$ is the n-fold tensor product, and the multiplication is given by the isomorphisms $B^{(n)} \otimes B^{(m)} \to B^{(n+m)}$ [DR75, p. 386]. By the $\mathbf{k}$-$\mathbf{k}$-bimodule structure, ${}_jB_i \otimes {}_lB_m = 0$ if $i \neq l$, so $B^{(n)} = 0$ if there is no path of length n in (Γ, Λ). Since (Γ, Λ) has no oriented cycles, the length of paths is bounded, so $\mathrm{T}(\mathbf{k}, B)$ is a finite dimensional k-algebra and we denote it by $k(\Gamma, \Lambda)$. Let $e_i \in \mathbf{k}$ be the n-tuple that has $1 \in \mathbf{k}_i$ in the ith place and 0 elsewhere. A left $k(\Gamma, \Lambda)$-module M is *finite-dimensional* if $\dim_{\mathbf{k}_i} e_i M < \infty$ for all i, which is equivalent to $\dim_k M < \infty$. We let f.d. $k(\Gamma, \Lambda)$ denote the category of finite dimensional left $k(\Gamma, \Lambda)$-modules. The categories $\mathrm{Rep}(\Gamma, \Lambda)$ and f.d. $k(\Gamma, \Lambda)$ are equivalent [DR75, Proposition 10.1] and we view the equivalence as an identification. Since (Γ, Λ) has no oriented cycles, $k(\Gamma, \Lambda)$ is hereditary [ARS97, III Proposition 1.4], meaning that all submodules of projective modules are projective. Further, by [GR92, Example 2 p. 88], f.d. $k(\Gamma, \Lambda)$ (and hence $\mathrm{Rep}(\Gamma, \Lambda)$) is an Abelian category. In this dissertation all $k(\Gamma, \Theta)$-modules are finite dimensional.

If (Γ, Λ) is a valued quiver and $x \in \Gamma_0$, let $\sigma_x\Lambda$ be the orientation on Γ obtained by reversing the direction of each arrow incident with x and preserving the directions of the remaining arrows. This generates a new valued quiver $(\Gamma, \sigma_x\Lambda)$ (with the same valuation $\mathbf{b}$ and modulation $\mathfrak{B}$). A vertex x is a *sink* if no arrow starts at x. For a sink x, we recall the definition of the *reflection* functor $F_x^+ : \mathrm{Rep}(\Gamma, \Lambda) \to \mathrm{Rep}(\Gamma, \sigma_x\Lambda)$ [BGP73, Definition 1.1 1)] and [DR76, pp. 15-16].

Let $(V, f) \in \mathrm{Rep}(\Gamma, \Lambda)$ and set $(W, g) = F_x^+(V, f)$. Then $W_y = V_y$ for all $y \neq x$, and $g_b = f_b$ for the arrows b of $(\Gamma, \sigma_x\Lambda)$ that do not start at x. Let $a_i : y_i \to x,\ i = 1, \ldots, l$, be the arrows of (Γ, Λ) ending at x. Then the reversed arrows $a_i' : x \to y_i$ are the arrows of $(\Gamma, \sigma_x\Lambda)$ starting at x. Consider the exact sequence

$$0 \to \mathrm{Ker}\, h \xrightarrow{j} \bigoplus_{i=1}^{l} {}_xB_{y_i} \otimes_{\mathbf{k}_{y_i}} V_{y_i} \xrightarrow{h} V_x$$

of $\mathbf{k}_x$-spaces where h is induced by the maps $f_{a_i} : {}_xB_{y_i} \otimes_{\mathbf{k}_{y_i}} V_{y_i} \to V_x$. Then $W_x = \mathrm{Ker}\, h$ and each map $g_{a_i'} : {}_{y_i}B_x \otimes_{\mathbf{k}_x} W_x \to W_{y_i} = V_{y_i}$ is obtained from the map $W_x \to {}_xB_{y_i} \otimes_{\mathbf{k}_{y_i}} W_{y_i}$ induced by j using the isomorphisms below [DR76, pp. 14-15].

$$\mathrm{Hom}_{\mathbf{k}_x}(W_x, {}_xB_{y_i} \otimes_{\mathbf{k}_{y_i}} W_{y_i}) \cong \mathrm{Hom}_{\mathbf{k}_x}(W_x, \mathrm{Hom}_{\mathbf{k}_{y_i}}({}_{y_i}B_x, \mathbf{k}_{y_i}) \otimes_{\mathbf{k}_{y_i}} W_{y_i}) \cong$$

$$\mathrm{Hom}_{\mathbf{k}_x}(W_x, \mathrm{Hom}_{\mathbf{k}_{y_i}}({}_{y_i}B_x, W_{y_i})) \cong \mathrm{Hom}_{\mathbf{k}_{y_i}}({}_{y_i}B_x \otimes_{\mathbf{k}_x} W_x, W_{y_i})$$

We finish this section by quoting part of [BGP73, Theorem 1.1 1)] and [DR76, Proposition 2.1].

Theorem 2.1. *Let x be a sink (respectively, source) in (Γ, Λ) and let $M \in \mathrm{f.d.}\, k(\Gamma, \Lambda)$ be an indecomposable module. If M is not simple projective (respectively, injective), then F_x^+M (respectively, F_x^-M) is indecomposable and $F_x^-F_x^+M \cong M$ (respectively, $F_x^+F_x^-M \cong M$)*

2.2 Admissible Sequences

A sequence of vertices $S = x_1, x_2, \ldots, x_s,\ s \geq 0$, is called *(+)-admissible* ([BGP73, p. 23]) on (Γ, Λ) if it either is empty, or satisfies the following conditions: x_1 is a sink in (Γ, Λ), x_2 is a sink in $(\Gamma, \sigma_{x_1}\Lambda)$, x_3 is a sink in $(\Gamma, \sigma_{x_2}\sigma_{x_1}\Lambda)$, $\ldots$, x_s is a sink in $(\Gamma, \sigma_{x_{s-1}} \ldots \sigma_{x_2}\sigma_{x_1}\Lambda)$. We denote by $\mathfrak{S}$ the set of (+)-admissible sequences on (Γ, Λ). If $S = x_1, \ldots, x_s$ is in $\mathfrak{S}$, we set $\Lambda^S = \sigma_{x_s} \cdots \sigma_{x_1}\Lambda$ and $F(S) = F^+_{x_s} \cdots F^+_{x_1} : \mathrm{Rep}(\Gamma, \Lambda) \to \mathrm{Rep}(\Gamma, \Lambda^S)$. If $T = u_1, \ldots, u_p, v_1, \ldots, v_q$ is (+)-admissible on (Γ, Λ), then $U = u_1, \ldots, u_p$ is (+)-admissible on (Γ, Λ) and $V = v_1, \ldots, v_q$ is (+)-admissible on (Γ, Λ^U); we write $T = UV$. A sequence $K \in \mathfrak{S}$ is *complete* if it contains each vertex $x_i \in \Gamma_0$ exactly once. If $K \in \mathfrak{S}$ is complete, Λ^K is the orientation obtained from Λ be reflecting each arrow exactly twice, so $\Lambda^K = \Lambda$. If $m > 0$ and K^m denotes the concatenation of m copies of K, we have $K^m \in \mathfrak{S}$.

Example 2.2. With the quiver defined in Example 2.1, the sequence $U = 3, 2, 4$ is (+)-admissible, and (preserving the valuation) the quiver is

$$(\Gamma, \sigma_4\sigma_2\sigma_3\Lambda) = (\Gamma, \Lambda^U) = \begin{array}{ccccc} 1 & \longleftarrow & 2 & \longrightarrow & 3 \\ & & \uparrow & & \\ & & 4 & & \end{array}$$

where the arrow $1 \to 2$ has been reflected once, while the arrows $2 \to 3$ and $4 \to 2$ have been reflected twice. The sequence $V = 1, 3, 2$ is (+)-admissible on (Γ, Λ^U), so the sequence $T = UV = 3, 2, 4, 1, 3, 2$ is (+)-admissible on (Γ, Λ).

We quote [KT05, Definition 1.2].

Definition 2.1. If $S = x_1, \ldots, x_i, x_{i+1}, \ldots, x_s,\ 0 < i < s$, is in $\mathfrak{S}$ and no edge of Γ connects x_i with x_{i+1}, then $T = x_1, \ldots, x_{i+1}, x_i, \ldots, x_s$ is in $\mathfrak{S}$ and we set SrT. Let $\sim$ be the equivalence relation that is the reflexive and transitive closure of the symmetric binary relation r.

It follows from the analog of [BGP73, Lemma 1.2, proof of part 3)] for representations of valued quivers that $S \sim T$ implies $F(S) = F(T)$. If $K \in \mathfrak{S}$ is complete, then $F(K) = \Phi^+$ is the *Coxeter* functor (Φ^+ does not depend on the choice of K) [DR76, p. 19]. To see that Φ^+ does not depend on the choice of K we follow the proof of [BGP73, Lemma 1.2 part 3)]. Let $H = h_1, h_2, \ldots, h_n$ and $K = k_1, k_2, \ldots, k_n$ be two distinct complete (+)-admissible sequences and let $k_1 = h_m$. Since k_1 is a sink in (Γ, Λ), the vertices $h_1, h_2, \ldots, h_{m-1}$ are not joined by an edge to k_1, so $H \sim k_1, h_1, h_2, \ldots, h_{m-1}, h_{m+1}, \ldots, h_n$. Carrying out a similar agrement with the vertices k_2, then k_3, and so on, we get $H \sim K$, so $F(H) = F(K)$. We say that $S \in \mathfrak{S}$ *annihilates* $M \in \text{f.d.}\, k(\Gamma, \Lambda)$ if $F(S)(V, f) = 0$ where $(V, f) \in \text{Rep}(\Gamma, \Lambda)$ is identified with M. In light of this identification, we often write $F(S)M$ or Φ^+M.

A *source* is a vertex of a quiver at which no arrow ends. Replacing sinks with sources, one gets similar definitions of a *reflection* functor F_x^-, a $(-)$-*admissible* sequence, and the *Coxeter* functor Φ^- [BGP73, DR76].

Following [DR76], we define preprojective representations as in [BGP73]. We quote [BGP73, Definition 1.3]

Definition 2.2. A module $M \in \text{f.d.}\, k(\Gamma, \Lambda)$ is *preprojective* if $(\Phi^+)^\nu M$ for some $\nu > 0$.

By [BGP73, Note 2], Definition 2.2 is equivalent to the following.

Definition 2.3. A module $M \in \text{f.d.}\, k(\Gamma, \Lambda)$ is *preprojective* if there exists an $S \in \mathfrak{S}$ that annihilates it.

While [BGP73, DR76] focused on Definition 2.2, it was Definition 2.3 that [KT05, KT06] used to explore the relation between representations (V, f) and $(+)$-admissible sequences. This dissertation extends and broadens the exploration started in [KT05, KT06].

2.3 A Special Case: Ordinary Quivers

The considerations of Sections 2.1 and 2.2 become simpler in the case of ordinary quivers. In this case, the valuation $\mathbf{b}$ is taken to be symmetric; $b_{ij} = b_{ji}$. In ordinary quivers, we allow multiple arrows between vertices, so the integer b_{ij} represents the number of arrows between vertex i and vertex j. In the definition of a representation of an ordinary quiver, the division rings $\mathbf{k}_i$, $i \in \Gamma_0$, of the modulation $\mathfrak{B}$ are taken to be a common field, k, and the bimodules ${}_iB_j$ and ${}_jB_i$ are powers of the field ${}_iB_j = {}_jB_i = k^{b_{ij}}$. Conditions (i) and (ii) are trivially satisfied. Notice that for an arrow of the valued quiver, $a : i \to j$, we have ${}_jB_i \otimes_{\mathbf{k}_i} V_i = k^{b_{ij}} \otimes_k V_i \simeq V_i^{b_{ij}}$, so the map $f_a : {}_jB_i \otimes_{\mathbf{k}_i} V_i \to V_j$ is isomorphic to a map $\bigoplus_{l=1}^{b_{ij}} V_i \xrightarrow{h} V_j$

Under such a modulation, a representation of an ordinary quiver without oriented cycles reduces to that given in [BGP73], which is the following. A *representation* (V, f) of an ordinary quiver (Γ, Λ) over k is a set of finite-dimensional k-spaces $\{V_x \,|\, x \in \Gamma_0\}$ together with k-linear maps $f_a : V_x \to V_y$ for each arrow $a : x \to y$.

The reflections σ_x and the functors $F_x^+ : \mathrm{Rep}(\Gamma, \Lambda) \to \mathrm{Rep}(\Gamma, \sigma_x\Lambda)$ also reduce in the case of ordinary quivers (see [BGP73]). If $(W, g) = F_x^+(V, f)$, let $a_i : y_i \to x,\ i = 1, \ldots, l$, be the arrows of (Γ, Λ) ending at x, then the reversed arrows $a_i' : x \to y_i,\ i = 1, \ldots, l$, are all the arrows of $(\Gamma, \sigma_x\Lambda)$ starting at x. The exact sequence defining W_x becomes the exact sequence

$$0 \to \mathrm{Ker}\, h \xrightarrow{j} \bigoplus_{i=1}^{l} V_{y_i} \xrightarrow{h} V_x$$

of k-spaces, where the map h is induced by the maps $f_{a_i} : V_{y_i} \to V_x$. As in the valued quiver case, $W_x = \mathrm{Ker}\, h$ and the maps $g_{a_i'} : W_x \to W_{y_i} = V_{y_i}$ are induced by j.

In the case of ordinary quivers, the algebra $k(\Gamma, \Lambda)$ also takes on an equivalent definition. We quote [Rin84] and [ARS97]. The path algebra, denoted $k(\Gamma, \Lambda)$, is the k-vector space with basis the set of all paths in (Γ, Λ). The product of two paths, p, q is given by $qp = \begin{cases} 0, & \text{if } e(p) = s(q) \\ 1, & \text{if } e(p) \neq s(q) \end{cases}$. By definition, we have a path e_x of length 0 for each $x \in \Gamma_0$, so $\sum_{x\in\Gamma_0} e_x$ is the identity in $k(\Gamma, \Lambda)$.

The categories f.d. $k(\Gamma, \Lambda)$ and $\mathrm{Rep}(\Gamma, \Lambda)$ are again equivalent, and we sketch the equivalence of objects of these categories as given in [ARS97, III Theorem 1.5]. Define $F : \mathrm{Rep}(\Gamma, \Lambda) \to$ f.d. $k(\Gamma, \Lambda)$ by $F(V, f) = \amalg_{i\in\Gamma_0} V_i$. If $\pi_x : \bigoplus_{i\in\Gamma_0} V_i \to V_x$ is projection and $\zeta_x : V_x \to \bigoplus_{i\in\Gamma_0} V_i$ is inclusion, then the module structure is induced by $a.F(V,f) = \zeta_y f_a \pi_x(\bigoplus_{i\in\Gamma_0} V_i)$ for $a : x \to y$. Conversely, define $G : \text{f.d.}\, k(\Gamma,\Lambda) \to \text{Rep}(\Gamma,\Lambda)$ by $G(M) = (V,f)$ where $V_x = e_x.M$ and $f_a = \pi_y a \zeta_x : V_x \to V_y$ as induced by the composition $V_x \xrightarrow{\zeta_x} \text{f.d.}\, k(\Gamma,\Lambda) \xrightarrow{a} \text{f.d.}\, k(\Gamma,\Lambda) \xrightarrow{\pi_y} V_y$.

Chapter 3

Lattice of Admissible Sequences

3.1 An Equivalence Relation on Sequences

We begin by recalling some of the results of [KT05] needed in this dissertation.

Definition 3.1. For $S = x_1, \ldots, x_s$ in $\mathfrak{S}$ with $s \geq 0$, we define the *length* of S as $\ell(S) = s$; the *support* of S as $\operatorname{Supp} S = \{v \in \Gamma_0 \mid \exists j, 0 < j \leq s, \text{with } v = x_j\}$; and for all $v \in \Gamma_0$, the *multiplicity* of v in S, $m_S(v)$, as the (nonnegative) number of subscripts j satisfying $x_j = v$.

The following statement, which is [KT05, Proposition 1.9], gives a canonical form in $\mathfrak{S}$.

Proposition 3.1. *Let $S \in \mathfrak{S}$ be nonempty.*

(a) We have $S \sim S_1 S_2 \cdots S_r$, the concatenation of $S_1, \ldots, S_r$, where, for all i, S_i consists of distinct vertices, and for $i < r$, $\operatorname{Supp} S_i \subseteq \operatorname{Supp} S_{i+1}$. Further, if

$\operatorname{Supp} S_i \neq \Gamma_0$ *then* $\operatorname{Supp} S_{i+1} \subsetneq \operatorname{Supp} S_i$.

(b) Let $T \sim T_1 T_2 \cdots T_q$ *be a nonempty sequence in* $\mathfrak{S}$ *where, for all* j, T_j *consists of distinct vertices, and for* $j < q$, $\operatorname{Supp} T_j \subseteq \operatorname{Supp} T_{j+1}$. *Then* $S \sim T$ *if and only if* $r = q$ *and* $S_i \sim T_i$ *on* $(\Gamma, \Lambda^{S_1 \cdots S_{i-1}}), i = 1, \ldots, r$.

For $S \in \mathfrak{S}$, the sequence $S_1 S_2 \cdots S_r$ of Proposition 3.1(a) is called the *canonical form* of S, the integer r is the *size* of S, and S_i is the ith *segment* of S.

Remark 3.1. In the setting of Proposition 3.1(a), if $v \in \Gamma_0$ then $v \in \operatorname{Supp} S_i$ if and only if $m_S(v) \geq i$.

We use the following example to illustrate the above definitions:

Example 3.1. For the quiver in Example 2.1, the sequence $T = 3, 2, 4, 1, 3, 2$ in Example 2.2 has length $\ell(T) = 6$, support $\operatorname{Supp} T = \{1, 2, 3, 4\}$, and the multiplicity of 3 is $m_T(3) = 2$. Further, T is already written in canonical form, with $T_1 = 3, 2, 4, 1$ and $T_2 = 3, 2$. Since 1 and 3 are not connected by an edge, $T \sim 3, 2, 4, 3, 1, 2$. The latter sequence is not in canonical form.

The sequences in $\mathfrak{S}$ are classified up to equivalence in terms of filters of (Γ_0, Λ). Recall that a subset F of a poset $(P, \leq)$ is a *filter* if for all $x \in F$ and $y \in P$, $x \leq y$ implies $y \in F$; a filter F is *principal* if $F = \langle x \rangle = \{y \in P \mid x \leq y\}$. For a filter F of (Γ_0, Λ), the *hull* of F, $H_\Lambda(F)$, is the smallest filter of (Γ_0, Λ) containing F and each vertex of Γ_0 connected by an edge to a vertex in F.

Remark 3.2. If F is a filter of (Γ_0, Λ) such that the full subgraph of Γ determined by $\operatorname{Supp} F$ is connected (for example, if F is a principal filter), then the full subgraph of Γ determined by $\operatorname{Supp} H_\Lambda(F)$ is connected.

If $X \subset \Gamma_0$, there exists an $S \in \mathfrak{S}$ satisfying $\operatorname{Supp} S = X$ if and only if X is a filter of (Γ_0, Λ), and if $\operatorname{Supp} S = X$ and the vertices of S are distinct, then S is unique up to equivalence [KT05, Proposition 1.3]. We now recall the classification of sequences in $\mathfrak{S}$ given in [KT05, Proposition 1.11].

Proposition 3.2. (a) *Let $S = S_1 S_2 \cdots S_r \in \mathfrak{S}$ be a nonempty sequence in canonical form. Then, for all i, $\operatorname{Supp} S_i$ is a filter of (Γ_0, Λ) and, for $0 < i < r$, $H_\Lambda(\operatorname{Supp} S_{i+1}) \subset \operatorname{Supp} S_i$.*

(b) *Let $F_1 \supset \cdots \supset F_{r-1} \supset F_r$ be a sequence of nonempty filters of (Γ_0, Λ) satisfying $H_\Lambda(F_{i+1}) \subset F_i$ for $0 < i < r$. Then there exists a sequence $S_1 S_2 \cdots S_r \in \mathfrak{S}$ in canonical form, unique up to equivalence, satisfying $\operatorname{Supp} S_i = F_i$ for all i.*

3.2 Ordering Admissible Sequences

The following is [KT05, Definition 2.1].

Definition 3.2. If $S, T \in \mathfrak{S}$, we say that S is a *subsequence* of T and write $S \preccurlyeq T$ if $T \sim SU$ for some (+)-admissible sequence U.

It was shown in [KT05, Section 2] that $\preccurlyeq$ is a preorder on $\mathfrak{S}$, and that $S \preccurlyeq T$ and $T \preccurlyeq S$ if and only if $S \sim T$. Therefore the preorder $\preccurlyeq$ induces a partial order on the set of equivalences classes of sequences in $\mathfrak{S}$. We often identify equivalent (+)-admissible sequences and then say that $\preccurlyeq$ is a partial order on $\mathfrak{S}$. The next statement is a characterization of the preorder in terms of the canonical form.

Proposition 3.3. *Let $S, T \in \mathfrak{S}$ be nonempty and let $S_1 \cdots S_r$, $T_1 \cdots T_q$ be their canonical forms, respectively. Then the following are equivalent.*

(a) $S \preccurlyeq T$.

(b) $r \leq q$ *and* $\operatorname{Supp} S_i \subset \operatorname{Supp} T_i$ *for* $0 < i \leq r$.

(c) For all $v \in \Gamma_0$, $m_S(v) \leq m_T(v)$.

Proof. The fact that $S \preccurlyeq T$ if and only if $r \leq q$ and $S_i \preccurlyeq T_i$ for $0 < i \leq r$ is [KT05, Proposition 2.1(c)]. Since $S_i, T_i \in \mathfrak{S}$ consist of distinct vertices, $S_i \preccurlyeq T_i$ is equivalent to $\operatorname{Supp} S_i \subset \operatorname{Supp} T_i$ according to [KT05, Proposition 1.6, parts (a) and (b)]. Therefore (a) is equivalent to (b).

The fact that (b) and (c) are equivalent is an immediate consequence of Remark 3.1. □

Corollary 3.4. *For $S, T \in \mathfrak{S}$, $S \sim T$ if and only if for all $v \in \Gamma_0$, $m_S(v) = m_T(v)$.*

The relations $\sim$ and $\preccurlyeq$ satisfy the left cancellation property.

Proposition 3.5. *Let $S \in \mathfrak{S}$ and let U, V be (+)-admissible sequences on $\left(\Gamma, \Lambda^S\right)$.*

(a) $SU \preccurlyeq SV$ *if and only if* $U \preccurlyeq V$.

(b) $SU \sim SV$ *if and only if* $U \sim V$.

Proof. Part (a) is an immediate consequence of the equivalence of parts (a) and (c) of Proposition 3.3, and (b) follows directly from Corollary 3.4. □

3.3 A Lattice Structure

To show that the poset of equivalence classes of (+)-admissible sequences is a lattice, we define the greatest lower and the least upper bounds, $\wedge$ and $\vee$.

Definition 3.3. Let $S, T \in \mathfrak{S}$ be nonempty and let $S_1 S_2 \cdots S_r$, $T_1 T_2 \cdots T_q$ be their canonical forms, respectively, where without loss of generality we assume that $r \leq q$. We set:

(a) $S \wedge T$ to be a (+)-admissible sequence with the canonical form $R_1 R_2 \cdots R_r$ where $\operatorname{Supp} R_i = \operatorname{Supp} S_i \cap \operatorname{Supp} T_i$ for $0 < i \leq r$.

(b) $S \vee T$ to be a (+)-admissible sequence with the canonical form $R_1 R_2 \cdots R_q$ where $\operatorname{Supp} R_i = \operatorname{Supp} S_i \cup \operatorname{Supp} T_i$ for $0 < i \leq r$, and $\operatorname{Supp} R_i = \operatorname{Supp} T_i$ for $r < i \leq q$.

If $\emptyset$ is the empty sequence in $\mathfrak{S}$, then for all $S \in \mathfrak{S}$, we set $S \wedge \emptyset = \emptyset$ and $S \vee \emptyset = S$.

Remark 3.3. Definition 3.3 contains two claims that must be supported. The first claim is that $S \wedge T$ and $S \vee T$ are (+)-admissible sequences. Secondly, implicit in the definition is that $S \wedge T$ and $S \vee T$ are well defined up to equivalence. Both of these statements are proven in the following proposition.

Proposition 3.6. *The poset of equivalence classes of* $\sim$ *in* $\mathfrak{S}$ *with the partial order* $\preccurlyeq$ *is a lattice where the operations of the greatest lower bound and the least upper bound are* $\wedge$ *and* $\vee$*, respectively.*

Proof. The intersection or union of two filters is always a filter. If F_1, F_2 are filters of (Γ_0, Λ), then it is straightforward that $H_\Lambda(F_1 \cap F_2) \subset H_\Lambda(F_1) \cap H_\Lambda(F_2)$ and $H_\Lambda(F_1 \cup F_2) = H_\Lambda(F_1) \cup H_\Lambda(F_2)$. Therefore, in view of Proposition 3.2, we conclude that if $S, T \in \mathfrak{S}$, then $S \wedge T$ and $S \vee T$ are in $\mathfrak{S}$. By Proposition 3.3, any sequence $R \preccurlyeq S$ and $R \preccurlyeq S$ must have $\operatorname{Supp} R_i \subseteq \operatorname{Supp} S_i \cap \operatorname{Supp} T_i$ for admissible i, so $\operatorname{Supp} R_i \subseteq \operatorname{Supp}(S \wedge T)i$. This gives $R \preccurlyeq S \wedge T$, so $S \wedge T$ is a greatest lower bound. Proposition 3.1 (b) gives that this lower bound is unique up to equivalence. In a similar way, $S \vee T$ is shown to be the least upper bound, unique up to equivalence. □

Example 3.2. Continuing with the quiver from Example 2.1, the sequences $S = 3, 2, 3$ and $T = 3, 2, 4, 1$ are (+)-admissible with $S \wedge T \sim 3, 2$ and $S \vee T \sim 3, 2, 4, 1, 3$.

The following statement is a generalization of [KT05, Lemma 1.4]

Theorem 3.7. *Let* $S, T \in \mathfrak{S}$ *be nonempty.*

(a) $S \sim (S \wedge T)\, U$ *and* $T \sim (S \wedge T)\, V$*, where* U *and* V *are (+)-admissible sequences on* $\left(\Gamma, \Lambda^{S \wedge T}\right)$ *that are unique up to equivalence.*

(b) $\operatorname{Supp} U \cap \operatorname{Supp} V = \emptyset$.

(c) UV *and* VU *are (+)-admissible sequences on* $\left(\Gamma, \Lambda^{S \wedge T}\right)$ *and* $UV \sim VU$*.*

(d) $S \vee T \sim (S \wedge T)\, UV \sim SV \sim TU$.

Proof. (a) This is a direct consequence of Propositions 3.6 and 3.5(b).

(b) By (a), we have $(S \wedge T)(U \wedge V) \preccurlyeq S, T$, so Proposition 3.6 implies $(S \wedge T)(U \wedge V) \preccurlyeq S \wedge T$ therefore $U \wedge V = \emptyset$. By Definition 3.3(a) and Proposition 3.1(a), $\operatorname{Supp} U \cap \operatorname{Supp} V = \emptyset$.

(c) Since $\operatorname{Supp} U$ is a filter, there is no arrow $u_i \to v_j$ in (Γ, Λ) with $u_i \in \operatorname{Supp} U$, $v_j \in \operatorname{Supp} V$, and a similar conclusion holds for $\operatorname{Supp} V$. Now the statement follows immediately from (b).

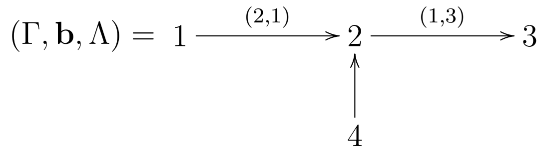
(d) By (a) and Proposition 3.6, we have $S \vee T \sim (S \wedge T)\, UV' \sim (S \wedge T)\, VU'$, for some U', V', as well as $S, T \preccurlyeq (S \wedge T)UV \sim (S \wedge T)VU$, using (c). By Proposition 3.6,

$$S \vee T \sim (S \wedge T)\, UV' \sim (S \wedge T)\, VU' \preccurlyeq (S \wedge T)UV \sim (S \wedge T)VU.$$

Applying the cancellation laws of Proposition 3.5 to the displayed formulas, we get $UV' \sim VU'$ and $V' \preccurlyeq V, U' \preccurlyeq U$. By (b), $m_{U'}(u) = m_U(u)$ for $u \in \operatorname{Supp} U$ and $m_{V'}(v) = m_V(v)$ for $v \in \operatorname{Supp} V$, so Corollary 3.4 implies $U' \sim U$ and $V' \sim V$. Thus (d) holds. □

We end this chapter with an example illustrating Theorem 3.7 (a) and a remark about the lattice of admissible sequences.

Example 3.3. Recall the quiver of Example 2.1.



Let $S = 3, 2, 3$ and $T = 3, 2, 4, 1$. Then $S \wedge T \sim 3, 2$, and $S \vee T \sim 3, 2, 4, 1, 3$ as in Example 3.2. Then by Theorem 3.7 Part(a), $S \sim (S \wedge T)\, U$, $T \sim (S \wedge T)\, V$ for sequences $U \sim 3$ and $V \sim 4, 1$.

The next fact was noted by Speyer in [Spe06].

Remark 3.4. A lattice is *distributive* if the meet distributes over the join [Grä71, p. 36], which is to say $S \wedge (T \vee U) = (S \wedge T) \vee (S \wedge U)$. This is equivalent to the join distributing over the meet, $S \vee (T \wedge U) = (S \vee T) \wedge (S \vee U)$ [Grä71, Section 4 Lemma 10]. Although it is not used explicitly in this thesis, it is easy to verify that the lattice of (+)-admissible sequences is a distributive lattice. This follows from the facts that the i^{th} segment of $S \wedge T$ is defined by $S_i \cap T_i$ and the i^{th} segment of $S \vee T$ is defined by $S_i \cup T_i$. The fact that the lattice of sets, under the operations $\cap$ and $\cup$, is a distributive lattice gives that the lattice of (+)-admissible sequences is a distributive lattice.

Chapter 4

Principal Admissible Sequences

4.1 Constructing Sequences

We quote [KT05, Definition 2.2].

Definition 4.1. A nonempty sequence $S \in \mathfrak{S}$ is *principal* if its canonical form $S_1 S_2 \cdots S_r$ satisfies $\operatorname{Supp} S_i = H_\Lambda(\operatorname{Supp} S_{i+1})$ for $0 < i < r$ and $\operatorname{Supp} S_r$ is a principal filter. We denote by $\mathfrak{P}$ the set of principal sequences in $\mathfrak{S}$. By Proposition 3.2, a principal sequence is determined uniquely up to equivalence by its size r and the set $\operatorname{Supp} S_r$, so we let $S_{r,x}$ denote the principal sequence of size r with $\operatorname{Supp} S_r = \langle x \rangle,\ x \in \Gamma_0$. Thus $\mathfrak{P} = \{S_{r,x} \,|\, r \in \mathbb{Z}^+, x \in \Gamma_0\}$ where $\mathbb{Z}^+$ is the set of positive integers.

Remark 4.1. It follows from Remark 3.2 that if $S \in \mathfrak{P}$, the full subgraph of Γ determined by $\operatorname{Supp} S$ is connected.

We quote [KT05, Corollary 2.3].

Proposition 4.1. *Let $S, T \in \mathfrak{S}$ be nonempty, let $S_1 \cdots S_r$ be the canonical form of S, and let $T = y_1, \ldots, y_t$ be in $\mathfrak{P}$. If $T \sim S_{q,y}$ then:*

(a) $T \preccurlyeq S$ if and only if $q \leq r$ and $y \in \operatorname{Supp} S_q$.

(b) $y_t = y$.

A nonempty sequence in $\mathfrak{S}$ is the join of some sequences in $\mathfrak{P}$.

Proposition 4.2. *Let $\emptyset \neq S \in \mathfrak{S}$ and let $S_1 \cdots S_r$ be the canonical form of S. Set $S_{r+1} = \emptyset$ and $\operatorname{Supp} S_{r+1} = \emptyset$.*

(a) $S = \bigvee_{(h,v)} S_{h,v}$ where $0 < h \leq r$ and, for each h, v runs through the set of minimal elements of $\operatorname{Supp} S_h \setminus H_\Lambda(\operatorname{Supp} S_{h+1})$ in the partial order of (Γ_0, Λ).

(b) If $S = T_1 \vee \cdots \vee T_l$ where $T_i \in \mathfrak{P}$ for all i, then for each pair (h, v) described in (a), there exists an i satisfying $S_{h,v} \sim T_i$.

(c) There exist $T_1, \ldots, T_l \in \mathfrak{P}$ satisfying $S = T_1 \vee \cdots \vee T_l$. If l is the smallest possible and $S = U_1 \vee \cdots \vee U_l$ where $U_1, \ldots, U_l \in \mathfrak{P}$, there exists a reindexing so that $T_i \sim U_i$ for all i.

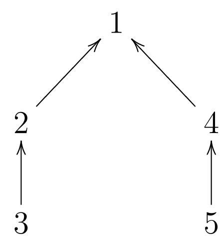
Proof. (a) Proceed by induction on r. If $r = 1$, then $h = 1$ in all pairs (h, v) and $\operatorname{Supp} S = \operatorname{Supp} S_1$ is a filter of (Γ_0, Λ). Since a nonempty filter is the union of the principal filters generated by its minimal elements, the statement follows from Definition 3.3(b). Suppose now that $r > 1$ and the statement holds for all nonempty sequences in $\mathfrak{S}$ of size $< r$. By the induction hypothesis, $S_2 \cdots S_r = \bigvee_{(h,v)} S_{h-1,v}$ where $1 < h \leq r$. It follows from Definition 3.3(b) that $S = (\bigvee_{(1,v)} S_{1,v}) \bigvee (\bigvee_{(h,v)} S_{h,v})$ where $1 < h \leq r$. If $u \in \operatorname{Supp} S_1 \setminus H_\Lambda(\operatorname{Supp} S_2)$ satisfies $u < v$ in the poset (Γ_0, Λ), then $S_{1,v} \preccurlyeq S_{1,u}$ in $\mathfrak{S}$. Therefore $\bigvee_{(1,v)} S_{1,v} = \bigvee_{(1,u)} S_{1,u}$ where u runs through the set of minimal elements of $\operatorname{Supp} S_1 \setminus H_\Lambda(\operatorname{Supp} S_2)$. The proof of (a) is complete.

(b) Suppose $S = T_1 \vee \cdots \vee T_l$ where $T_i \in \mathfrak{P}$ for all i. Since $v \in \operatorname{Supp} S_h$ for each (h, v), Definition 3.3(b) implies that there exists an i such that the canonical form $W_1 \cdots W_q$ of T_i satisfies $v \in \operatorname{Supp} W_h$. Thus $h \leq q$ and Proposition 4.1(a) says that $S_{h,v} \preccurlyeq T_i$. Since $T_i \in \mathfrak{P}$, we have $T_i \sim S_{p,u}$ for some $p > 0$ and $u \in \Gamma_0$, therefore $u \in \operatorname{Supp} S_p$. Therefore there exists a pair (j, w), where w is a minimal element of $\operatorname{Supp} S_j \setminus H_\Lambda(\operatorname{Supp} S_{j+1})$, such that the canonical form $X_1 \cdots X_j$ of $S_{j,w}$ satisfies $u \in \operatorname{Supp} X_p$. Then $p \leq j$ and $T_i \preccurlyeq S_{j,w}$ therefore $S_{h,v} \preccurlyeq S_{j,w}$. By Proposition 4.1(a), $h \leq j$ and $v \in \operatorname{Supp} X_h$. If $h < j$, then $v \in \operatorname{Supp} X_h = H_\Lambda(\operatorname{Supp} X_{h+1}) \subset H_\Lambda(\operatorname{Supp} S_{h+1})$, which contradicts the conditions imposed on the pair (h, v) in (a). Therefore we must have $h = j$. Since $S_{h,v} \preccurlyeq S_{h,w}$, then $\langle v \rangle \subset \langle w \rangle$ and $w \leq v$. The latter implies $w = v$ because v, w are minimal elements of $\operatorname{Supp} S_h \setminus H_\Lambda(\operatorname{Supp} S_{h+1})$. It follows that $T_i \sim S_{h,v}$.

(c) The statement is a consequence of (a) and (b). $\square$

The following example illustrates Proposition 4.2.

Example 4.1. Consider the following quiver:



The sequence $S = 1, 2, 3, 4, 5, 1, 2, 3 \in \mathfrak{S}$ is in canonical form, with $S_1 = 1, 2, 3, 4, 5$ and $S_2 = 1, 2, 3$. Also, $\operatorname{Supp} S_1 \backslash H_\Lambda(\operatorname{Supp} S_2) = \{5\}$, $\operatorname{Supp} S_2 = \{1, 2, 3\}$ with minimal element 3, so the proposition gives $S \sim S_{1,5} \vee S_{2,3} = 4, 2, 5 \vee 1, 2, 3, 4, 1, 2, 3$.

4.2 Shortest Annihilating Sequences

We quote [KT06, Definition 2.1].

Definition 4.2. If $S \in \mathfrak{S}$ annihilates a $k(\Gamma, \Lambda)$-module M, but no proper subsequence of S annihilates M, we call S a *shortest* sequence annihilating M.

The following statement quotes [KT06, Proposition 2.1 and Theorems 2.2 and 2.6].

Theorem 4.3. *Let M be a preprojective $k(\Gamma, \Lambda)$-module.*

(a) Up to equivalence, there exists a unique shortest sequence $S_M \in \mathfrak{S}$ annihilating M.

(b) If M is indecomposable and N is an indecomposable preprojective $k(\Gamma, \Lambda)$-module, then $S_N \sim S_M$ if and only if $N \cong M$.

(c) *If M is indecomposable, then $S_M \in \mathfrak{P}$.*

(d) *If M is indecomposable and $S_M = x_1, \ldots, x_s$, then $M \cong F^-_{x_1} \cdots F^-_{x_{s-1}}(L_{x_s})$ where L_{x_s} is the simple projective $k(\Gamma, \sigma_{x_{s-1}} \cdots \sigma_{x_1} \Lambda)$-module associated with $x_s \in \Gamma_0$.*

We now drop the assumption of indecomposability of M in the description of S_M.

Theorem 4.4. *Let M be a preprojective $k(\Gamma, \Lambda)$-module.*

(a) *Up to equivalence, there exists a unique shortest sequence $S_M \in \mathfrak{S}$ annihilating M. If $M \cong M_1^{m_1} \oplus \cdots \oplus M_t^{m_t}$ where the M_i's are nonisomorphic indecomposable $k(\Gamma, \Lambda)$-modules and $m_i > 0$ for all i, then each M_i is preprojective and $S_M = S_{M_1} \vee \cdots \vee S_{M_t}$.*

(b) *If L is a preprojective $k(\Gamma, \Lambda)$-module, then $S_L \preccurlyeq S_M$ if and only if for each indecomposable direct summand X of L, there exists an indecomposable direct summand Y of M satisfying $S_X \preccurlyeq S_Y$.*

Proof. (a) If $M = 0$ then $S_M = \emptyset$. If $M \neq 0$, then $M \cong M_1^{m_1} \oplus \cdots \oplus M_t^{m_t}$ as indicated in the statement. Since every reflection functor is additive, each M_i is preprojective. By Theorem 4.3(a), a sequence $S \in \mathfrak{S}$ annihilates M if and only if $S_{M_i} \preccurlyeq S$ for all i. Since $\mathfrak{S}$ is a lattice by Proposition 3.6, we have $S_M = S_{M_1} \vee \cdots \vee S_{M_t}$. Alternatively, we note that the proof of the corresponding statement for ordinary quivers (see [KT05, pp. 394-395]), does not use the indecomposability of M and works for any nonzero preprojective M.

(b) The statement is trivial if either L or M is zero. Assuming L, M are nonzero, write $L \cong L_1^{l_1} \oplus \cdots \oplus L_s^{l_s}$ and $M \cong M_1^{m_1} \oplus \cdots \oplus M_t^{m_t}$ as in (a). For the sufficiency, suppose that for each i there exists a j satisfying $S_{L_i} \preccurlyeq S_{M_j}$. By (a), $S_{L_i} \preccurlyeq S_M$ therefore $S_L = S_{L_1} \vee \cdots \vee S_{L_s} \preccurlyeq S_M$ because $\mathfrak{S}$ is a lattice according to Proposition 3.6. For the necessity, let $T_1 \cdots T_q$ be the canonical form of S_M and let $X = L_i$. By Theorem 4.3(c), $S_X \in \mathfrak{P}$, so that $S_X \sim S_{r,x}$ where $r > 0$ and $x \in \Gamma_0$. It is clear that $S_X \preccurlyeq S_L$, so $S_L \preccurlyeq S_M$ implies $S_X \preccurlyeq S_M$. By Proposition 4.1(a), $r \leq q$ and $x \in \operatorname{Supp} T_r$. By Definition 3.3(b), $\operatorname{Supp} T_r$ is the union of rth segments of some of the sequences $S_{M_1}, \ldots, S_{M_t}$. Thus, for some j, the canonical form of S_{M_j} is $U_1 \cdots U_p$ where $r \leq p$ and $x \in \operatorname{Supp} U_r$. By Proposition 4.1(a), $S_X \preccurlyeq S_{M_j}$. □

Remark 4.2. Part (b) of Theorem 4.3 is false without the assumption that both M and N are indecomposable. For example, if M is indecomposable and $N = L \oplus M$ where L is indecomposable preprojective with $S_L \preccurlyeq S_M$, then $S_M = S_N$ but $M \ncong N$.

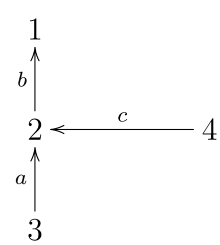
Since $\mathfrak{P}$ is a subset of $\mathfrak{S}$, the partial order $\preccurlyeq$ on the set of equivalence classes of $\sim$ in $\mathfrak{S}$ induces a partial order on the set of equivalence classes of $\sim$ in $\mathfrak{P}$. Identifying equivalent sequences in $\mathfrak{P}$, we often say that $\preccurlyeq$ is a partial order on $\mathfrak{P}$. The poset structure of $\mathfrak{P}$ carries a lot of information about the preprojective component of the Auslander-Reiten quiver of $k(\Gamma, \Lambda)$.

4.3 Quivers Related to (Γ, Λ)

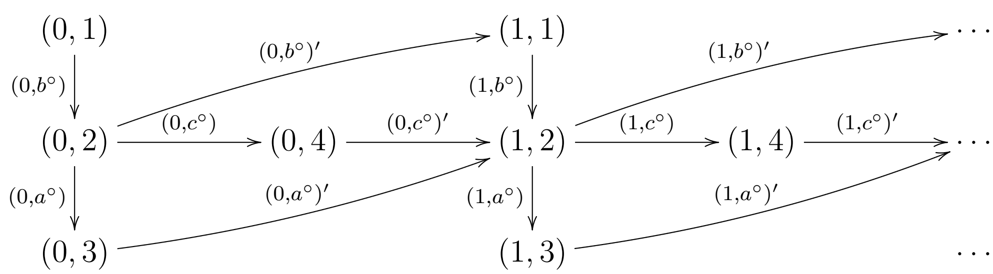
In this section, we follow [ARS97, Rin84] and construct two quivers from (Γ, Λ). The first is the translation quiver of the opposite quiver. The second is the preprojective component of the Auslander-Reiten Quiver.

We define the translation quiver as in [Rin84, p. 47]. Let $\mathbb{N}$ be the set of non-negative integers. The translation quiver $\mathbb{N}(\Gamma, \Lambda^{op})$ of the opposite quiver of (Γ, Λ) is defined as follows. The set of vertices of $\mathbb{N}(\Gamma, \Lambda^{op})$ is $\mathbb{N} \times \Gamma_0$, and each arrow $a : u \to v$ of (Γ, Λ) gives rise to two series of arrows, $(n, a^\circ) : (n, v) \to (n, u)$ and $(n, a^\circ)' : (n, u) \to (n + 1, v)$. The translation is defined as $\tau : \mathbb{N}\backslash\{0\} \times \Gamma_0 \to \mathbb{N} \times \Gamma_0$ given by the formula $\tau(n, v) = (n-1, v)$. By construction, $\mathbb{N}(\Gamma, \Lambda^{op})$ is a locally finite quiver without oriented cycles, so $\mathbb{N} \times \Gamma_0$ is a poset as explained earlier. We omit both the translation and the valuation when dealing with the translation quiver.

Example 4.2. We illustrate the translation quiver with the following example. Let (Γ, Λ) be the quiver from Example 2.1, written as



The translation quiver, $\mathbb{N}(\Gamma, \Lambda^{op})$ is the following.

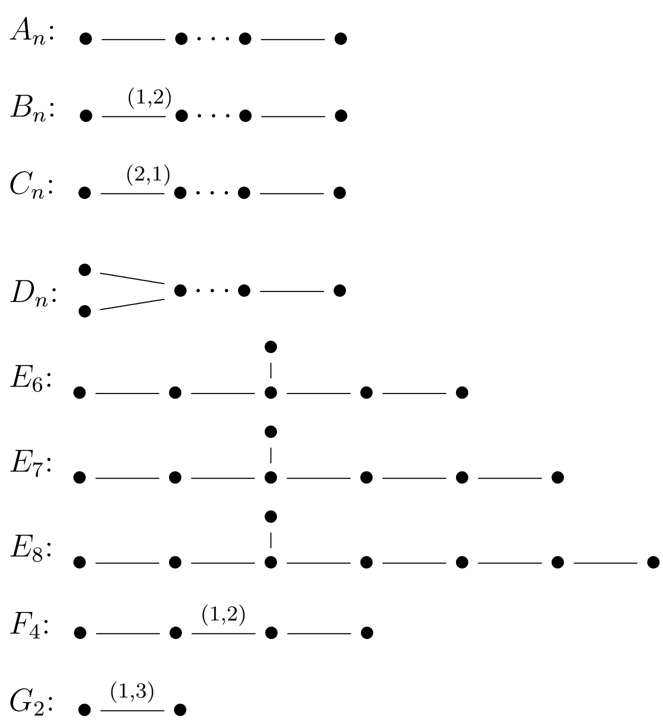


Let $X \in$ f.d. $k(\Gamma, \Lambda)$ be indecomposable and let $[X]$ be the isomorphism class of X. If $Y \in$ f.d. $k(\Gamma, \Lambda)$ is indecomposable, a path of length $m > 0$ from X to Y is a sequence of nonzero nonisomorphisms $X = A_0 \to \cdots \to A_m = Y$, where $A_i \in$ f.d. $k(\Gamma, \Lambda)$ is indecomposable for all i. By definition, there exists a path of length zero from X to X. One writes $[X] \prec [Y]$ if there exists a path of positive length from X to Y.

For $A, B, C, X \in$ f.d. $k(\Gamma, \Lambda)$ a morphism $f : B \to C$ is a *split epimorphism* if the identity on C factors through f, that is, there exists an $h : C \to B$ such that $1_C = fh$. Likewise, a morphism $g : A \to B$ is a *split monomorphism* if the identity on A factors through g, that is, there exists an $h : B \to A$ such that $1_A = hg$ [ARS97, Section V.1]. A morphism $f : B \to C$ is called *irreducible* if it is neither a split epimorphism nor a split monomorphism and if $g = ts$ for some $s : B \to X$ and $t : X \to C$, then either s is a split monomorphism or t is a split epimorphism [ARS97, Section V.5].

The preprojective component of (Γ, Λ), $\tilde{\mathfrak{P}}(\Gamma, \Lambda)$ (see [ARS97, Sections VII.1 and VIII.1]), is a locally finite connected quiver whose set of vertices, $\tilde{\mathfrak{P}}(\Gamma, \Lambda)_0$, consists of the isomorphism classes of indecomposable preprojective $k(\Gamma, \Lambda)$-modules, and the number of arrows $[X] \to [Y]$ is the k-dimension of the vector space $\mathrm{Irr}\,(X, Y)$ of irreducible maps $X \to Y$. If X, Y are indecomposable where Y is preprojective, and if $X = A_0 \to \cdots \to A_m = Y$, $m > 0$, is a path from X to Y, then $[X] \neq [Y]$ [ARS97, VIII Proposition 1.5] and A_i is preprojective for all i. It follows that the reflexive closure $\preccurlyeq$ of the transitive binary relation $\prec$ is a partial order on $\tilde{\mathfrak{P}}(\Gamma, \Lambda)_0$. Moreover, $[X] \prec [Y]$ if and only if there is a finite sequence of irreducible morphisms $X = B_0 \to \cdots \to B_n = Y$, where $n > 0$ and B_j is indecomposable preprojective for all j. For $M \in \mathrm{f.d.}\,k(\Gamma, \Lambda)$ an indecomposable preprojective, and m is the smallest integer such that $(\Phi^+)^m M = 0$, set $\nu(M) = m - 1$, and $x(M)$ to be the vertex of (Γ, Λ) associated with the indecomposable projective module $(\Phi^+)^\nu L \cong (\mathrm{DTr})^\nu L$.

Recall that the Dynkin diagrams are graphs of types A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4, or G_2.



Theorem 4.5. *(a) The map $\psi : \mathfrak{P} \to \mathbb{N} \times \Gamma_0$ given by $\psi(S_{r,x}) = (r-1, x)$ is an isomorphism of posets.*

(b) Consider the map $\phi : \tilde{\mathfrak{P}}(\Gamma, \Lambda) \to \mathbb{N}(\Gamma, \Lambda^{op})$ defined on the vertices by $\phi([L]) = (\nu, x) = (\nu(L), x(L))$, and defined on the arrows in a natural way [ARS97, VIII Proposition 1.15]. Given an $[M] \in \tilde{\mathfrak{P}}(\Gamma, \Lambda)_0$, the map ϕ induces a bijection between the set of paths in $\tilde{\mathfrak{P}}(\Gamma, \Lambda)$ ending at $[M]$ and the set of paths in $\mathbb{N}(\Gamma, \Lambda^{op})$ ending at $\phi([M])$.

(c) The map $\chi : \tilde{\mathfrak{P}}(\Gamma, \Lambda)_0 \to \mathfrak{P}$ given by $[L] \mapsto S_L$ is an injective morphism of posets.

(d) If Γ is not a Dynkin diagram, the maps ϕ and χ are isomorphisms.

Proof. (a) This is [KT05, Theorem 2.5(a)].

(b) This is [KT06, Proposition 2.8(d)].

(c) This is [KT06, Corollary 2.9(a)].

(d) This is [KT06, Proposition 2.8(b) and Corollary 2.9(c)] with [DR76, Theorem (a)] □

The next result will play a crucial role in Chapter 5.

Proposition 4.6. *If M, N are indecomposable preprojective $k(\Gamma, \Lambda)$-modules, then $[M] \preccurlyeq [N]$ in $\tilde{\mathfrak{P}}(\Gamma, \Lambda)_0$ if and only if $S_M \preccurlyeq S_N$ in $\mathfrak{P}$.*

Proof. The necessity is an immediate consequence of Theorem 4.5(c). We now assume that $S_M \preccurlyeq S_N$ and show that $[M] \preccurlyeq [N]$. If $S_N \preccurlyeq S_M$, then $S_M \sim S_N$ so Theorem

4.3(b) implies $M \cong N$ and $[M] = [N]$; in particular, $[M] \preccurlyeq [N]$. Suppose now that $S_N \not\preccurlyeq S_M$ where $S_M \sim S_{p,u}$ and $S_N \sim S_{q,v}$ for some $p, q > 0$ and $u, v \in \Gamma_0$. By Theorem 4.5(a), $(p-1, u) < (q-1, v)$ in $\mathbb{N} \times \Gamma_0$; therefore there is a path $(p-1, u) \to (q-1, v)$ of positive length in $\mathbb{N}(\Gamma, \Lambda^{op})$. By Theorem 4.5(b), there is a path $[M] \to [N]$ of positive length in $\tilde{\mathfrak{P}}(\Gamma, \Lambda)$, i.e., $[M] \preccurlyeq [N]$. □

4.4 A Special Case, Part II

We return to the study of ordinary quivers, and use this study to prove Proposition 4.8, which will be crucial to results in Chapter 5.

Proposition 4.7. *For* (Γ, Λ) *an ordinary quiver, Theorem 4.5 Part (d) reduces to saying that the maps* ϕ *and* χ *are isomorphisms if* Γ *is not a Dynkin diagram of the types A, D, or E.*

Proof. The statements [KT06, Proposition 2.8(b) and Corollary 2.9(c)] are generalizations of [KT05, Proposition 3.7(b) and Corollary 3.8(c)], which state that ϕ and χ are isomorphisms if (Γ, Λ) is of infinite representation type. In the case of ordinary quivers, Gabriel's Theorem (see [BGP73]) states that the quivers of finite representation type are exactly those where Γ is a Dynkin diagram of the types A, D, or E. □

Since ordinary quivers may have multiple edges, whereas this thesis requires that valued quivers only have single edges, Proposition 4.7 is used in the following crucial result.

Proposition 4.8. *For $S_{r,x} \sim S = x_1, \ldots, x_s,\ s > 1$, $T = x_2, \ldots, x_s$ is a principal (+)-admissible sequence on $(\Gamma, \sigma_{x_1} \Lambda)$. If $S_1 \cdots S_r$ and $T_1 \cdots T_q$ are the canonical forms of S and T, respectively, then $\operatorname{Supp} T_q$ is the principal filter of $(\Gamma_0, \sigma_{x_1} \Lambda)$ generated by x, and we have:*

(a) If $x_1 = x$, then $q = r - 1$ and $\operatorname{Supp} T_i = \operatorname{Supp} S_i$ for $0 < i < r$.

(b) If $x_1 \neq x$, then $q = r$, $\operatorname{Supp} T_i = \operatorname{Supp} S_i$ for $i \neq m_S(x_1)$, and $\operatorname{Supp} T_i = \operatorname{Supp} S_i \setminus \{x_1\}$ for $i = m_S(x_1)$.

Proof. Without loss of generality, we may assume that Γ is not a Dynkin diagram of the type A, D, or E. For if it is, there must be at least one arrow in (Γ, Λ) because Γ is a connected graph with more than one vertex. We double the arrow preserving its direction. The new graph is no longer a Dynkin diagram, but the new quiver has the same sets $\mathfrak{P}$ and $\mathfrak{S}$ as the original quiver.

By Theorem 4.7, the map χ of Theorem 4.5(c) is an isomorphism. Thus $S \sim S_M$ for some indecomposable preprojective $k(\Gamma, \Lambda)$-module M, and $T = S_{F^+_{x_1} M}$ where, by Theorem 2.1, $F^+_{x_1} M$ is an indecomposable preprojective $k(\Gamma, \sigma_{x_1} \Lambda)$-module because $s > 1$. By Theorem 4.3(c), T is a principal (+)-admissible sequence on $(\Gamma, \sigma_{x_1} \Lambda)$. Since $S_{r,x} \sim S$, Proposition 4.1(b) says that $x_s = x$ and $\operatorname{Supp} T_q$ is the principal filter of $(\Gamma_0, \sigma_{x_1} \Lambda)$ generated by x. We also have $m_T(v) = m_S(v)$ if $x_1 \neq v \in \Gamma_0$, and $m_T(x_1) = m_S(x_1) - 1$. Comparing the multiplicities of vertices in S and T, using

Remark 3.1, and taking into account that $\operatorname{Supp} S_r = \{x\}$ if $x_1 = x$, we see that (a) and (b) hold. □

The statement of Proposition 4.8 does not involve representation theory, and there is a purely combinatorial proof that is longer and more technical than the one given above. As noted in the proof, if $S \sim S_M$ where M is an indecomposable preprojective $k(\Gamma, \Lambda)$-module, then $T \sim S_{F^+_{x_1} M}$ where $F^+_{x_1} M$ is an indecomposable preprojective $k(\Gamma, \sigma_{x_1} \Lambda)$-module. Since an indecomposable preprojective module is uniquely determined up to isomorphism by the shortest (+)-admissible sequence that annihilates it (Theorem 4.3(b)), the explicit computation of the canonical form of T in terms of the canonical form of S allows one to think of a positive reflection functor as operating on principal (+)-admissible sequences instead of indecomposable preprojective modules. In particular, knowing the pair (r, x), which determines the location of M in the preprojective component of (Γ, Λ), one may compute the pair (q, x) that determines the location of $F^+_{x_1} M$ in the preprojective component of $(\Gamma, \sigma_{x_1} \Lambda)$ (see Theorem 4.5).

Chapter 5

Reduced words in the Weyl group

5.1 Two Types of Reflections

The following definition of the Weyl Group comes from [DR76]. For a graph Γ with a valuation $\mathbf{b}$ we assume for the rest of this dissertation that $\Gamma_0 = \{1, \ldots, n\}$. The matrix $A = (a_{ij})$ defined by $a_{ii} = 2$ and $a_{ij} = -b_{ij}$ for all $i \neq j$ is an $n \times n$ *generalized Cartan matrix.* As stated in Section 2.1, we assume that Γ is connected, so A is *indecomposable* in the sense that it is not conjugate under a permutation matrix to a block-diagonal matrix $\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$. Further, as demonstrated in Example 2.1, A is *symmetrizable* since there is a matrix D with $d_{ij} = 0$ for $i \neq j$ and $d_{ii} \neq 0$ for all i such that DA is a symmetric matrix. With this identification, to any finite connected valued graph without loops, there is a corresponding indecomposable symmetrizable generalized Cartan matrix, unique up to conjugation by a permutation matrix. Likewise, from any such matrix we can construct a corresponding valued graph.

The graph generates reflections $\sigma_i : \mathbb{Z}^n \to \mathbb{Z}^n$ defined by $\sigma_i(\mathbf{x}) = \mathbf{y}$, where $\mathbf{x}_k = \mathbf{y}_k$ for $k \neq i$ and $\mathbf{y}_i = -\mathbf{x}_i + \sum_{k \in \Gamma_0} b_{ki}\mathbf{x}_k$. The Weyl Group, $\mathcal{W}$, is the subgroup of $GL(\mathbb{Z}^n)$ generated by $\sigma_1, \ldots, \sigma_n$. We denote the i_{th} standard basis vector of $\mathbb{Z}^n$ by e_i. A vector $\mathbf{x}$ is said to be *positive*, written $\mathbf{x} > 0$, if $\mathbf{x} \neq 0$ and $\mathbf{x}_i \geq 0$ for all i. Similarly, a vector is *negative*, written $\mathbf{x} < 0$, if $\mathbf{x} \neq 0$ and $\mathbf{x}_i \leq 0$ for all i.

The following theorem is part of [BGP73, Theorem 1.1 1)] and [DR76, Proposition 2.1].

Theorem 5.1. *Let x be a sink (respectively, source) in (Γ, Λ) and $M \in \text{f.d.}\, k(\Gamma, \Lambda)$ an indecomposable module.*

(a) If M is not simple projective (respectively, injective), then

$\dim F_x^+ M = \sigma_x(\dim M)$ *(respectively,* $\dim F_x^- M = \sigma_x(\dim M)$*).*

(b) If M is simple projective (respectively, injective), then $F_x^+ M = 0$ (respectively, $F_x^- M = 0$) and $\sigma_x(\dim M) < 0$.

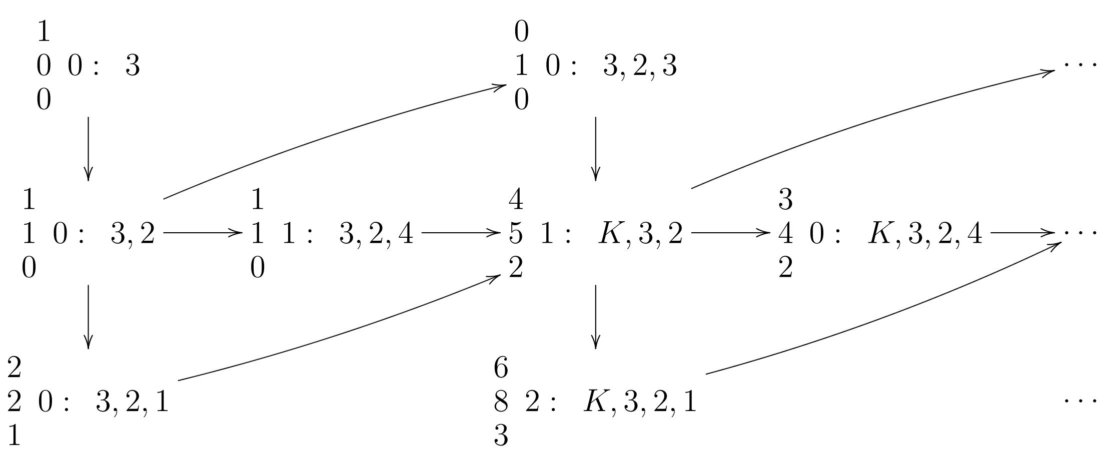
Definition 5.1. If $S = x_1, \ldots, x_s$ is in $\mathfrak{S}$, we set $w(S) = \sigma_{x_s} \cdots \sigma_{x_1}$ and say that $w(S)$ is the word in the Weyl group $\mathcal{W}$ associated to S. If no edge connects vertices i and j, then $\sigma_i \sigma_j = \sigma_j \sigma_i$ so that $S \sim T$ implies $w(S) = w(T)$.

To illustrate the utility of words associated to sequences in $\mathfrak{S}$, we begin with an elementary proof of the following well-known fact (see [ARS97, VIII Corollary 2.3]).

Proposition 5.2. *Let* $M, N \in \text{f.d.}\, k(\Gamma, \Lambda)$ *be indecomposable. If* M *is preprojective and* $\dim M = \dim N$*, then* $M \cong N$*.*

Proof. If $S_M = x_1, \ldots, x_s$ and $T = x_1, \ldots, x_{s-1}$, then $F(S_M)M = 0$ and $F(T)M \neq 0$. Using Theorems 2.1 and 5.1, we obtain $w(S_M)(\dim N) = w(S_M)(\dim M) < 0$. Using the same theorems, we get $F(S_M)N = 0$; therefore N is preprojective and $S_N \preccurlyeq S_M$. Since N is preprojective, by symmetry we get $S_M \preccurlyeq S_N$ therefore $S_M \sim S_N$. By Theorem 4.3(b), $M \cong N$. □

Example 5.1. The construction of $\tilde{\mathfrak{P}}(\Gamma, \Lambda)$, where each vertex is replaced by the dimension vector of the corresponding module, is described in [GR92, Section 10]. Theorem 4.5 and Proposition 5.2 give new tools for this construction, based on shortest annihilating sequences, with Theorem 5.1 giving an explicit construction for those dimension vectors. In this way, and writing $\dim M = \begin{matrix} \dim V(1) \\ \dim V(2) \;\; \dim V(4) \\ \dim V(3) \end{matrix} : S_M$, and recalling that K denotes a complete (+)-admissible sequence, the translation quiver of Example 4.2 becomes the following.



5.2 Indecomposable Modules and Reduced Words

Recall (see [Bou02]) that for $w \in \mathcal{W}$, the *length* of w, $\ell(w)$, is the smallest integer $l \geq 0$ such that w is the product of l simple reflections, and a word $w = \sigma_{y_t} \cdots \sigma_{y_1}$ in $\mathcal{W}$ is *reduced* if $\ell(w) = t$.

Remark 5.1. If $S \preccurlyeq T$ in $\mathfrak{S}$ where $w(T)$ is reduced, then $T \sim SU$ for some U, and $w(S), w(U)$ are reduced, as follows from $w(T) = w(U)w(S)$.

We examine the words in the Weyl group associated to preprojective modules.

Theorem 5.3. *Let M be a preprojective $k(\Gamma, \Lambda)$-module.*

(a) The word $w(S_M) \in \mathcal{W}$ is reduced.

(b) If M is indecomposable and N is an indecomposable preprojective $k(\Gamma, \Lambda)$-module, the following are equivalent.

(i) $M \cong N$.

(ii) $S_M \sim S_N$.

(iii) $w(S_M) = w(S_N)$.

Proof. (a) If $M = 0$ the statement is trivial. If $M \neq 0$, let $S_M = x_1, \ldots, x_s$ and proceed by induction on $s > 0$. The case $s = 1$ is clear, so suppose $s > 1$ and the statement holds for all orientations Θ on Γ without oriented cycles and all preprojective $k(\Gamma, \Theta)$-modules N for which $S_N = y_1, \ldots, y_t$ satisfies $t < s$. Since $s > 1$, $F^+_{x_1} M \neq 0$

is a preprojective $k(\Gamma, \sigma_{x_1} \Lambda)$-module, and $S_{F^+_{x_1} M} = x_2, \ldots, x_s$. By the induction hypothesis, the word $u = \sigma_{x_s} \cdots \sigma_{x_2}$ in $\mathcal{W}$ is reduced. Assume, to the contrary, that the word $u\sigma_{x_1} = \sigma_{x_s} \cdots \sigma_{x_2}\sigma_{x_1}$ is not reduced. Then $\ell(u\sigma_{x_1}) \leq \ell(u)$ and, since $\mathcal{W}$ is a Coxeter group [Kac90, Proposition 3.13], we must have $\ell(u\sigma_{x_1}) < \ell(u)$ [Bou02, Ch. IV, Proposition 1.5.4]. By [Kac90, Lemma 3.11, part a)], $\sigma_{x_s} \cdots \sigma_{x_2}(e_{x_1}) < 0$; therefore $F(S_{F^+_{x_1} M})L_{x_1} = 0$ where L_{x_1} is the simple $k(\Gamma, \sigma_{x_1} \Lambda)$-module associated to x_1, as follows from Theorems 2.1 and 5.1. Since x_1 is a sink in (Γ, Λ), it is a source in $(\Gamma, \sigma_{x_1} \Lambda)$ so L_{x_1} is a simple injective and preprojective $k(\Gamma, \sigma_{x_1} \Lambda)$-module. In particular, $[L_{x_1}]$ is a sink in $\tilde{\mathfrak{P}}(\Gamma, \sigma_{x_1} \Lambda)$ and, thus, a maximal element of the poset $\tilde{\mathfrak{P}}(\Gamma, \sigma_{x_1} \Lambda)_0$. On the other hand, $S_{L_{x_1}} \preccurlyeq S_{F^+_{x_1} M}$ so, by Theorem 4.4(b), $S_{L_{x_1}} \preccurlyeq S_N$ for some indecomposable direct summand N of $F^+_{x_1} M$ and, by Proposition 4.6, $[L_{x_1}] \preccurlyeq [N]$ in $\tilde{\mathfrak{P}}(\Gamma, \sigma_{x_1} \Lambda)_0$. Since $[L_{x_1}]$ is a maximal element, we have $[L_{x_1}] = [N]$ therefore $L_{x_1} \cong N$. This is in contradiction with the fact that the simple module associated to a vertex that is a source is not a direct summand of a module that belongs to the image of the positive reflection functor associated to the vertex, as follows from Theorem 2.1. Thus $w(S_M)$ is a reduced word.

(b) By Theorem 4.3(b), (i) is equivalent to (ii). It is clear that (ii) $\Longrightarrow$ (iii). To prove (iii) $\Longrightarrow$ (ii), suppose $w(S_M) = w(S_N)$. In view of parts (a) and (b) of Theorem 3.7, we have $S_M \sim (S_M \wedge S_N)U$ and $S_N \sim (S_M \wedge S_N)V$ where U, V are (+)-admissible sequences on $(\Gamma, \Lambda^{S_M \wedge S_N})$ satisfying $\operatorname{Supp} U \cap \operatorname{Supp} V = \emptyset$. If both U and V are empty, then $S_M \sim S_N$. If not, then, say, $U = u_1, \ldots, u_p$ with $p > 0$. We obtain

$w(U)w(S_M \wedge S_N) = w(V)w(S_M \wedge S_N)$; therefore $w(U) = w(V)$. By (a) and Remark 5.1, the word $w(U) = \sigma_{u_p} \cdots \sigma_{u_1}$ is reduced, so [Kac90, Lemma 3.11, part b)] says that $w(U)(e_{u_1}) < 0$. On the other hand, $w(V)(e_{u_1})$ is a root whose u_1-coordinate is the same as that of e_{u_1}, namely, is equal to 1, because $\operatorname{Supp} U \cap \operatorname{Supp} V = \emptyset$. Thus $w(V)(e_{u_1}) > 0$, which contradicts $w(U) = w(V)$. □

Zelevinsky suggested the following statement.

Corollary 5.4. *For $S = x_1, \ldots, x_s \in \mathfrak{S}$, $s > 0$, set $M(S) = F^-_{x_1} F^-_{x_2} \cdots F^-_{x_{s-1}}(L_{x_s})$.*

(a) If the word $w(S) \in \mathcal{W}$ is reduced, then $M(S)$ is an indecomposable module in $\tilde{\mathfrak{P}}$.

(b) If $M \in \tilde{\mathfrak{P}}$ is indecomposable, then $M \cong M(S)$ for some sequence $S \in \mathfrak{S}$ where $\ell(S) > 0$ and the word $w(S)$ is reduced.

Proof. (a) Since $w(S)$ is reduced, [Kac90, Lemma 3.10] implies that for $0 < i < s$, $\sigma_{x_i} \cdots \sigma_{x_{s-1}}(e_{x_s}) > 0$. By Theorems 2.1 and 5.1, $M(S)$ is an indecomposable $k(\Gamma, \Lambda)$-module and $F(S)(M(S)) = 0$. Thus $M(S) \in \tilde{\mathfrak{P}}$.

(b) By Theorems 4.3(d) and 5.3(a), $M \cong M(S_M)$ and $w(S_M)$ is reduced. □

In view of Theorem 4.3(c) the following result is a converse of Proposition 4.8.

Lemma 5.5. *Let $S = x_1, x_2, \ldots, x_s$, $s > 1$, be in $\mathfrak{S}$, suppose that the full subgraph of Γ determined by* Supp S *is connected, and set $T = x_2, \ldots, x_s$. If $T \sim S_N$ for some indecomposable preprojective $k(\Gamma, \sigma_{x_1} \Lambda)$-module N satisfying $M = F^-_{x_1} N \neq 0$, then M is indecomposable preprojective and $S \sim S_M$.*

Proof. By Theorem 2.1, M is indecomposable and $F^+_{x_1}F^-_{x_1}N \cong N$. Thus S annihilates M and $S_M \preccurlyeq x_1 \vee S_M \sim x_1U \preccurlyeq S = x_1T$ for some U, so by Proposition 3.5(a), $U \preccurlyeq T$. On the other hand, x_1U annihilates M, so U annihilates N, giving $T \preccurlyeq U$, thus $U \sim T$ and $x_1 \vee S_M \sim S$. Since the full subgraph of Γ determined by $\operatorname{Supp} S$ is connected, for some $x_l \in \operatorname{Supp} S \setminus \{x_1\}$ there is an arrow $x_l \to x_1$ in (Γ, Λ). This gives $x_l \in \operatorname{Supp} S_M$, so $x_1 \in \operatorname{Supp} S_M$, because $\operatorname{Supp} S_M$ is a filter of (Γ_0, Λ) by [KT06, Proposition 1.4(a)]. Thus $S_M \sim x_1 \vee S_M \sim S$. □

Theorem 5.6. *For $S = x_1, \ldots, x_s \in \mathfrak{P}$, $s > 0$, the following are equivalent.*

(a) There exists an indecomposable preprojective $k(\Gamma, \Lambda)$-module M satisfying $S \sim S_M$.

(b) The word $w(S) \in \mathcal{W}$ is reduced.

(c) For $0 < i < s$, $\sigma_{x_i} \cdots \sigma_{x_{s-1}}(e_{x_s}) > 0$.

Proof. (a) $\Longrightarrow$ (b) This is Theorem 5.3(a).

(b) $\Longrightarrow$ (c) This is addressed in the proof of Corollary 5.4(a).

(c) $\Longrightarrow$ (a) Proceed by induction on s. If $s = 1$ then $S \sim S_{L_{x_1}}$ where L_{x_1} is the simple projective $k(\Gamma, \Lambda)$-module associated to $x_1 \in \Gamma_0$. Suppose $s > 1$ and the statement holds for all principal (+)-admissible sequences of length $< s$ on all quivers (Γ, Θ) without oriented cycles. By Proposition 4.8, $T = x_2, \ldots, x_s$ is a principal (+)-admissible sequence on $(\Gamma, \sigma_{x_1}\Lambda)$, so by the induction hypothesis, $T \sim S_N$ for some indecomposable preprojective $k(\Gamma, \sigma_{x_1}\Lambda)$-module N, and Theorem 4.3(d) says that $N \cong F^-_{x_2} \cdots F^-_{x_{s-1}}(L_{x_s})$ where L_{x_s} is the simple projective $k(\Gamma, \sigma_{s-1} \cdots \sigma_{x_1} \Lambda)$-module associated to $x_s \in \Gamma_0$. Since $\sigma_{x_1} \cdots \sigma_{x_{s-1}}(e_{x_s}) > 0$, Theorem 5.1 implies that $M = F^-_{x_1} N \neq 0$. By Remark 4.1, the full subgraph of Γ determined by $\operatorname{Supp} S$ is connected. By Lemma 5.5, M is indecomposable and $S \sim S_M$. □

5.3 Admissible Sequences and Reduced Words

With the exception of Theorem 5.3 (a), Section 5.2 focused on indecomposable modules and the corresponding words in the Weyl group. We now turn our focus to the relation between admissible sequences and their corresponding words. The next theorem strengthens Theorem 5.6.

Theorem 5.7. *For all $S \in \mathfrak{S}$, the following are equivalent.*

(a) There exists a preprojective $k(\Gamma, \Lambda)$-module M satisfying $S \sim S_M$.

(b) The word $w(S) \in \mathcal{W}$ is reduced.

If $S \neq \emptyset$, let $S = T_1 \vee \cdots \vee T_l$ where, for all i, $T_i \in \mathfrak{P}$ (see Proposition 4.2). Then either of (a), (b) is equivalent to the following condition.

(c) For $0 < i \leq l$, the word $w(T_i) \in \mathcal{W}$ is reduced.

Proof. The case $S = \emptyset$ is clear, so let $S \neq \emptyset$.

(a) $\Longrightarrow$ (b) This is Theorem 5.3(a).

(b) $\Longrightarrow$ (c) Since $T_i \preccurlyeq S$, the statement follows from Remark 5.1.

(c) $\implies$ (a) By Proposition 4.2, we may take l to be as small as possible. By Theorem 5.6, $T_i \sim S_{M_i}$ for some indecomposable preprojective $k(\Gamma, \Lambda)$-module M_i. Since l is the smallest possible, for $i \neq j$, we have $T_i \not\sim T_j$ so that $M_i \ncong M_j$ by Theorem 4.3(b). By Theorem 4.4(a), $S \sim S_M$ where $M = M_1 \oplus \cdots \oplus M_l$. □

Remark 5.2. As mentioned in the introduction, Gabriel's Theorem established that for ordinary quivers of finite type (i.e. Γ is a Dynkin diagram of type A, D, E_6, E_7, or E_8) the indecomposable preprojective representations are in one-to-one correspondence with the positive roots of the Weyl group. Using the methods that Bernšteĭn, Gel'fand, and Ponomarev developed in [BGP73], Dlab and Ringel extend these results to (valued) quivers [DR76, Theorem]. Theorems 5.3 and 5.7 further extend these results to show that for any quiver, the indecomposable preprojective representations are in one-to-one correspondence with the elements of the Weyl group that have a reduced expression generated by a principal admissible sequences.

Example 5.2. Given a graph Γ and a reduced word $w \in \mathcal{W}$, it may be impossible to find an orientation Λ and a $(+)$-admissible sequence S of length $\ell(w)$ on (Γ, Λ) satisfying $w = w(S)$.

For example, if $\Gamma = A_4$:

$$x_1 \overset{a}{\text{——}} x_2 \text{——} x_3 \overset{b}{\text{——}} x_4 \ ,$$

then $w = \sigma_{x_2}\sigma_{x_3}\sigma_{x_2} = \sigma_{x_3}\sigma_{x_2}\sigma_{x_3}$ is reduced. If $w = w(S)$ where S is a $(+)$-admissible sequence of length 3 on (Γ, Λ) for some Λ, then either $S = x_2, x_3, x_2$ or $S = x_3, x_2, x_3$.

In the former case we must have $a : x_1 \to x_2$ in (Γ, Λ). Then in $(\Gamma, \sigma_{x_3}\sigma_{x_2}\Lambda)$ we have $a : x_2 \to x_1$ therefore x_2 is not a sink, a contradiction. If $S = x_3, x_2, x_3$, the argument is the same, using b instead of a.

Corollary 5.8. *Suppose* Γ *is not a Dynkin diagram and let* $S \in \mathfrak{S}$*.*

(a) The word $w(S) \in \mathcal{W}$ *is reduced.*

(b) There exists a preprojective $k(\Gamma, \Lambda)$*-module* M *satisfying* $S \sim S_M$*.*

Proof. (a) By assumption, the finite-dimensional algebra $k(\Gamma, \Lambda)$ is of infinite representation type (see [DR76]), therefore there exist infinitely many nonisomorphic indecomposable preprojective $k(\Gamma, \Lambda)$-modules M [ARS97, VIII Proposition 1.16] and, by Theorem 4.3(b), the corresponding sequences $S_M \in \mathfrak{S}$ are pairwise nonequivalent. Since the poset (Γ_0, Λ) is finite, Proposition 3.2 implies that for a given $m \geq 0$, there exists a sequence S_M whose canonical form $T_1 \cdots T_q$ satisfies $m \leq q$ and $T_i = K$ for $0 < i \leq m$ where K is a complete sequence. By Theorem 5.3(a), $w(S_M)$ is reduced therefore so is $w(K^m)$ according to Remark 5.1. For any $S \in \mathfrak{S}$, Proposition 3.3 implies $S \preccurlyeq K^r$ where r is the size of S. Using Remark 5.1 again, we see that $w(S)$ is reduced.

(b) This is an immediate consequence of (a) and Theorem 5.7. □

Remark 5.3. In view of Theorem 5.7 and Corollary 5.8, for a given $S \in \mathfrak{S}$ one may ask how to determine whether the word $w(S)$ is reduced, and if so, how to find a preprojective module M satisfying $S \sim S_M$. To handle these questions efficiently, one should write S as the join of the smallest possible number of sequences $T_i \in \mathfrak{P}$ as explained in Proposition 4.2; verify that each $w(T_i)$ is reduced using Theorem 5.6(c); and set M to be the direct sum of M_i's, where M_i is the indecomposable preprojective $k(\Gamma, \Lambda)$-module obtained from T_i according to Theorem 4.3(d).

We now characterize infinite Weyl groups in terms of reduced words.

Recall that $1, \ldots, n$ are the distinct vertices of Γ and if $v_1, \ldots, v_n$ is a permutation of those vertices, then $c = \sigma_{v_n} \cdots \sigma_{v_1}$ is a *Coxeter element* of $\mathcal{W}$ (a *Coxeter transformation* in [DR76, p. 8]); c depends on the choice of the permutation $v_1, \ldots, v_n$ of the vertices $1, \ldots, n$.

Theorem 5.9. *Let $A = (a_{ij})$ be an indecomposable symmetrizable generalized $n \times n$ Cartan matrix, and let $c = \sigma_{v_n} \cdots \sigma_{v_1}$ be a Coxeter element of the Weyl group $\mathcal{W}$. Then $\mathcal{W}$ is infinite if and only if for all $m \in \mathbb{Z}$, $\ell(c^m) = |m|n$.*

Proof. The sufficiency is clear. For the necessity, note that there exists a unique orientation Λ on Γ for which the quiver (Γ, Λ) has no oriented cycles and $K = v_1, \ldots, v_n$ is a $(+)$-admissible sequence on (Γ, Λ) [DR76, p. 8]. Then $c = w(K)$ and $c^m = w(K^m)$ for all $m \geq 0$. Since $\mathcal{W}$ is infinite, [DR76, Theorem (a) and Proposition 1.5] say that Γ is not a Dynkin diagram. By Corollary 5.8(a), c^m is a reduced word. □

Chapter 6

Admissible Sequences and Coxeter Groups

6.1 Coxeter-sortable elements

The following definition quotes [Rea05, pp. 7-8].

Definition 6.1. Fix an arbitrary Coxeter element $c = \sigma_{v_n} \cdots \sigma_{v_1}$ in any Coxeter group $\mathcal{W}$ and write a half-infinite sequence of vertices

$$c^{\infty} = v_n, v_{n-1}, \ldots, v_1, v_n, v_{n-1}, \ldots, v_1, v_n, v_{n-1}, \ldots, v_1, \ldots$$

The *c-sorting word* for $w \in \mathcal{W}$ is the lexicographically first subsequence $v_{i_1}, \ldots, v_{i_s}$ of c^{∞} for which $\sigma_{v_{i_1}} \cdots \sigma_{v_{i_s}}$ is a reduced word for w. The c-sorting word can be interpreted as a sequence of subsets of Γ_0 by rewriting

$$c^{\infty} = v_n, v_{n-1}, \ldots, v_1 | v_n, v_{n-1}, \ldots, v_1 | v_n, v_{n-1}, \ldots, v_1 | \ldots$$

where the symbol "|" is called a *divider*. The subsets in the sequence are the sets of vertices of the c-sorting word that occur between adjacent dividers. This sequence contains a finite number of nonempty subsets, and if any subset is empty, then every later subset is also empty. An element $w \in \mathcal{W}$ is *c-sortable* if its c-sorting word defines a sequence of subsets that is decreasing under inclusion.

The next proposition shows how the material of the Chapter 5 applies to c-sortable elements.

Proposition 6.1. *Let (Γ, Λ) be a (connected) valued quiver, let $K = v_1, \ldots, v_n$ be a complete (+)-admissible sequence on (Γ, Λ), and let $S \in \mathfrak{S}$.*

(a) $c = \sigma_{v_n} \cdots \sigma_{v_1}$ is a Coxeter element of the Weyl group $\mathcal{W}$ of Γ.

(b) If $S \sim S_M$ for some preprojective $k(\Gamma, \Lambda)$-module M, then $w(S)^{-1}$ is a c-sortable element of $\mathcal{W}$.

Proof. (a) This is clear.

(b) If $S = x_1, \ldots, x_s$, then $S^t = x_s, \ldots, x_1$ is a $(-)$-admissible sequence with respect to a suitable orientation, and $w(S)^{-1} = w(S^t) = \sigma_{x_s} \cdots \sigma_{x_1}$. By Theorem 5.7, the word $w(S)$ is reduced, thus so is $w(S)^{-1}$. By Proposition 3.1, $S \sim S_1 S_2 \cdots S_r$ where each S_i consists of distinct vertices and $\operatorname{Supp} S_{i+1} \subset \operatorname{Supp} S_i$. Then $w(S^t) = w(S_1^t) \cdots w(S_r^t)$ where $\operatorname{Supp} S_i^t \supset \operatorname{Supp} S_{i+1}^t$. □

6.2 Weak Bruhat Order

The definition of $\preccurlyeq$ is similar to the definition of the *Weak Bruhat Order*, $\leq_{wb}$. We quote [BB05, Definition 3.1.1(i)].

Definition 6.2. We write $u \leq_{wb} w$ to mean that $w = us_1s_2\cdots s_k$ for some s_i simple reflections, such that $l(us_1s_2\cdots s_i) = l(u) + i$ for $0 \leq i \leq k$.

These two orders, however, are incomparable (meaning $u \leq_{wb} w \not\Rightarrow u \preccurlyeq w$ and $u \preccurlyeq w \not\Rightarrow u \leq_{wb} w$) as demonstrated by the following example.

Example 6.1. Let $(\Gamma, \Lambda) = x_1 \longrightarrow x_2 \longleftarrow x_3 \longleftarrow x_4$. From the definition, $x_2, x_3 \leq_{wb} x_2, x_3, x_2$, but the word $x_2, x_3, x_2 \notin \mathfrak{S}$ as shown in Example 5.2. To see that the other comparison does not hold, we change the orientation to $(\Gamma, \Lambda') = x_1 \longleftarrow x_2 \longleftarrow x_3 \longleftarrow x_4$ and notice that $K, x_1, x_2, x_3 \preccurlyeq KK$ (recall that K is a complete (+)-admissible sequence, in this case $K = x_1, x_2, x_3, x_4$). Since there is no indecomposable preprojective module M with $S_M = KK$, the word $w(KK)$ is not reduced, hence $l(KK) = l(K, x_1, x_2, x_3) - 1$. By the definition of the weak Bruhat order, $K, x_1, x_2, x_3 \not\leq_{wb} KK$.

6.3 Coxeter Groups

As noted in the introduction, after Speyer read the contents of this dissertation, he felt that he could emulate the proof of Theorem 5.9 in a purely group theoretic context. Following our proof, he translated the language of (+)-admissible sequences into that of group theory. The key step is to view reduced words in terms of inversions, rather than preprojective modules. In that way, he removed the requirement that $\mathcal{W}$ be obtained from a quiver. In this way, he proves the following result.

Theorem 6.2. *A Coxeter group $\mathcal{W}$ is infinite if and only if all powers of a Coxeter element are reduced.*

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