RECITATION 5

A *linear combination* of the vectors $\vec{v}_1,\ \vec{v}_2,\ \vec{v}_3,\ \ldots,\ \vec{v}_n$ is the sum

$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 + \ \ldots\ + c_n\vec{v}_n$$

for any constants $c_1,\ c_2,\ c_3,\ \ldots\ c_n$, where vectors $\vec{v}_1,\ \ldots\ \vec{v}_n$ are all in the same vector space. For example, in R^3 the vector $< -4, 13, -10 >$ is a linear combination of the vectors $< 1, 2, 1 >$ and $< 2, -3, 4 >$, since

$$< -4, 13, -10 >= 2\ < 1, 2, 1 > -3\ < 2, -3, 4 >$$

I discovered this by writing $< -4, 13, -10 >$ as an unknown linear combination of the two vectors

$$< -4, 13, -10 >= c_1\ < 1, 2, 1 > +c_2\ < 2, -3, 4 >$$

setting the first, second and third components of each side equal, and solving the simultaneous equations

$$-4 = 1c_1 + 2c_2 \qquad 13 = 2c_1 - 3c_2 \qquad -10 = 1c_1 + 4c_2.$$

1) (a) Find if vector $< 3, 2 >$ is a linear combination of vectors $< 1, 2 >$ and $< 2, 5 >$ by the strategy above.

(b) Find if vector $< 2, 0, 6 >$ is a linear combination of vectors $< 1, 0, 1 >$, $< 1, 1, 0 >$, $< 7, -3, 10 >$.

A set of vectors $\vec{v}_1,\ \vec{v}_2,\ \vec{v}_3,\ \ldots,\ \vec{v}_n$ is *linearly independent* if none of the vectors in the set can be written as a linear combination of the other vectors. If any of the vectors in the set CAN be written as a linear combination of the other vectors, the set is said to be *linearly dependent.* For example, from the paragraph above, it is clear that $< -4, 13, -10 >,\ < 1, 2, 1 >$ and $< 2, -3, 4 >$ do not form a linearly independent set in R^3 (or we can say they do form a linearly dependent set).

How can we check if a set of vectors is linearly independent in R^n? For example, how should we check if $< 1, 2, 3, 4 >,\ < 2, 1, -1, 2 >,\ < 3, 1, 1, 2 >$ and $< 3, 1, 4, 1 >$ form a linearly independent set in R^4? We could check if the first vector is a linear combination of the other three, but if the answer is no, we would still have to check if the second vector is a linear combination of the rest, and if still no, then if the third vector is a linear combination of the rest, etc. It is much more efficient to use a trick: check how a linear combination of all of them can add up to the zero vector by solving

$$c_1\ < 1, 2, 3, 4 > +c_2\ < 2, 1, -1, 2 > +c_3\ < 3, 1, , 1, 2 > +c_4\ < 3, 1, 4, 1 >=< 0, 0, 0, 0 >$$

If all the c_i's have to be zero, the set is linearly independent. If it is possible for any of the c_i's not to be zero, then the set is linearly dependent. Obviously, these problems can be tedious to solve, but it can be very important to determine if a set of vectors is linearly independent or dependent.

2) Determine if each of these sets of vectors in R^3 is linearly independent or linearly dependent.

a) $< 3, 4 >,\ < 2, -1 >$ in R^2

b) $< 2, 1, 4 >,\ < -3, 0, 5 >,\ < 9, 3, 7 >$ in R^3

Note: When you are checking for linear independence by this method, you will always find either all the c_i's are zero (when independent), or that there is an infinity of possible values for the c_i's (when dependent).

3) Suppose the vectors $\vec{u},\ \vec{v},$ and $\vec{w}$ are mutually perpendicular:

$$\vec{u}\cdot\vec{v} = \vec{u}\cdot\vec{w} = \vec{v}\cdot\vec{w} = 0$$

and suppose the vector $\vec{\psi}$ is a linear combination of $\vec{u},\ \vec{v}$, and $\vec{w}$:

$$\vec{\psi} = c_1\vec{u} + c_2\vec{v} + c_3\vec{w}$$

a) Find a formula for c_1. (Hint: use dot products)

b) What would be the formulas for c_2 and c_3?

Note: These neat formulas only work when the vectors $\vec{u}, \vec{v}, \vec{w}$ are perpendicular to each other. This, in fact, is the main reason that coordinate axes are chosen perpendicular to each other.

HINTS FOR INSTRUCTORS

(1) The algebra required for these problems is all fairly simple except for problem 2(b). It probably is not worthwhile to try to teach Gauss elimination if they do not know it. I would let them use any high school method. However, be aware that when there is an infinity of solutions, you can expect them to get somewhat confused.

(2) The idea of checking for a linear combination is probably more obvious for students than the method given for checking linear independence, which may require reinforcement from you. I urge you not to introduce any tricks (taking the determinant of n vectors in R^n, etc.)

(3) You might give them a hint for problem 3(a), namely: "Try dotting both sides of the equations with $\vec{u}$."

(4) I expect them to find this recitation fairly short. You might think about adding a problem. Obviously, you will have to think about this ahead of time. After the recitation we can all compare experiences.