RECITATION 4

Recall that the Taylor series about $x = 0$ of $\sin x,\ \cos x,\ e^x$ are

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \dots$$

$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \dots$$

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \dots$$

1) (a) Find an expansion for e^{it} by substituting it for x in the Taylor series for e^x.

(b) Using the fact that $i^2 = -1,\ i^3 = -i,\ i^4 = 1$, rearrange the expansion found in (a) to express e^{it} as the sum of strictly real terms plus i times another sum of strictly real terms.

(c) Looking at the Taylor series above, can you write e^{it} as the sum of a cosine term and a sine term? This is one way to see where Euler's formula comes from (although Euler himself used differential equations to make his discovery).

(d) Now redo parts (a),(b),(c) for e^{-it} (ie., start by substituting $-it$ for x in the Taylor series for e^x) to obtain Euler's formula for e^{-it} in terms of $\cos t$ and $\sin t$.

2) (a) Can you turn Euler's formulas around to obtain formulas for $\cos t$ and for $\sin t$ in terms of e^{it} and e^{-it}? (Hint: Try adding and subtracting the two Euler's formulas together)

(b) Replace t in each of these formulas with iy. You should end up with formulas for $\cos(iy)$ and $\sin(iy)$. This is how mathematicians can make sense of the sine or cosine of an imaginary number.

(c) In more advanced trig classes, $\cosh y$ (the *hyperbolic cosine function*) is defined to be $\dfrac{e^y + e^{-y}}{2}$ and $\sinh y$ (the *hyperbolic sine function*) is defined to be $\dfrac{e^y - e^{-y}}{2}$. Looking at your result from part (b), write $\cos iy$ and $\sin iy$ in terms of $\cosh y$ and $\sinh y$.

(d) Without using a calculator find $\sin(i \ln 2)$ numerically.

Note: *These formulas not only explain what the sine and cosine functions should equal for imaginary arguments, but they also explain why the hyperbolic <u>sine</u> and hyperbolic <u>cosine</u> functions, which are given entirely in terms of e^x and e^{-x}, have trig names.*

3. If you have time, use the sine sum formula, $\sin(A + B) = \sin A \cos B + \cos A \sin B$, to find $\sin(2 + 3i)$. Congratulations: you can evaluate trig functions at complex numbers, not just pure imaginary numbers.

HINTS FOR INSTRUCTORS

(1) You can help with the simplification of terms containing i^2, i^3, i^4 in 1(b), but give students time to figure out the rest. There should be sufficient time. Obviously, if individual groups are not getting it, give them help.

(2) Remind students again of the important fact that $\log x$ and $\ln x$ almost always mean the same thing in college level mathematics.

(2) You might wish to tell your students that complex values of trig functions come up in sophisticated engineering applications, for example, when a technique leads to the real axis being mapped into another curve in the complex plane. You might also point out that once the sine and cosine can be evaluated for complex numbers, so can the other trig functions, such as the tangent, secant, etc.