RECITATION 2

The *Taylor Series about* $x = 0$ (also called Macclaurin series) of any function $f(x)$ is the infinite sum:

$$f(0) + f'(0)\,x + \frac{1}{2!}f''(0)\,x^2 + \frac{1}{3!}f'''(0)\,x^3 + \frac{1}{4!}f''''(0)\,x^4 + \dots \tag{1}$$

The dots indicate that the sum goes on forever. This sum can be written in the suggestive notation $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}\,x^n$, where $f^{(n)}$ indicates the n^{th} derivative of f and $f^{(0)} = f$. Note that one can think of this, loosely speaking, as an infinite polynomial, since all the expressions in the sum are constants times powers of x. The importance of the Taylor series is the fact that for any function $f(x)$ which is *smooth* at $x = 0$, the Taylor series will exactly equal the function in an interval about $x = 0$. The full Taylor series will be dealt with in MATH 2224. However, in this recitation we shall compute only the first few terms of a Taylor series. The sum of these terms will be a good approximation to the function $f(x)$ for values of x sufficiently close to $x = 0$.

1) (a) Compute the zeroth through the fourth term of the Taylor series about $x = 0$ for $f(x) = e^x$. Do this by making a table whose columns are labeled:

n	$f^{(n)}(x)$	$f^{(n)}(0)$

with rows $n = 0,\ n = 1$, etc. Calling the sum of these terms $S(x)$, ie.,

$$S(x) = f(0) + \frac{f'(0)}{1}x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f''''(0)}{4!}x^4,$$

write out the polynomial $S(x)$. This is an approximation for $f(x) = e^x$.

(b) Compute $S(x)$ at $x = 1$. What is the percent error (as percentage of correct value for $f(x)$) in using $S(x)$ to approximate $f(x)$ at $x = 1$? Next compute $S(x)$ at $x = 2$ and the percent error (as percentage of correct value for $f(x)$). Is the Taylor series approximation getting worse?

2) (a) Compute the zeroth through the fifth term (ie., up to the x^5 term) of the Taylor series about $x = 0$ for $f(x) = \sin x$. Write out this approximation for $\sin x$.

(b) Compute the zeroth through the fifth term (ie., up to the x^5 term) of the Taylor series about $x = 0$ for $f(x) = \cos x$. Write out this approximation for $\cos x$.

Note: The three Taylor series you have so far computed are the ones most often known by memory by *good* engineers.

(c) Compare the exact value of $\cos x$ at $x = \frac{\pi}{4}$ and the value your approximation gives. What is the percentage error?

(d) Can you look at the first few terms for the Taylor series of $\cos x$ and guess what the infinite sum would look like? How about the infinite sum for $\sin x$?

(e) If you have the time, compute the zeroth through the fourth term of the Taylor series about $x = 0$ for $f(x) = \frac{1}{1-x}$. The infinite sum for this one is called a *geometric series.*

HINTS FOR INSTRUCTORS

(1) In class do not refer to infinite sums as *series*, despite the phrase *Taylor series.* Always call them *infinite sums.*

(2) We suggest writing the equation labeled (1) on the blackboard, along with a sketch of the table (but with entries only in the n-column).

(3) You might want to emphasize at some point that the infinite sum is exactly equal to the function, and that this is one way a pocket calculator might compute the sine and cosine functions.

(4) The “If you have the time” phrase means they should not feel obligated to finish that part at home, if not done in class.

(3) Remember, keep lecture down to about 5 minutes. This will take some advance planning.