Math 1224  \hspace{1cm} \textbf{Vector Geometry}

\textbf{RECITATION 9: Variable Acceleration}

\textbf{Goals:} The problems in this unit follow up on topics we explored in the last unit. In these problems, however, the acceleration comes from other sources besides gravity, and therefore may not be constant.

\textbf{Introduction}

A particle with a mass of 1 g and a charge of +2 C enters an electric field at the origin \((0, 0, 0)\) with an initial velocity of \((0, -2, 1)\) cm/s. If the electric field has coordinates \(E(t) = (4 \cos t, 4 \sin t, 0)\) dynes/C, find the particle's trajectory.

\textbf{Solution:} Because the particle is charged, the electric field exerts a 'pull' on the particle, causing it to accelerate according to the formula

\[
\ddot{a} = \frac{q}{m} \vec{E},
\]

where \(q\) and \(m\) denote the charge and mass of the particle, respectively.

So the particle's acceleration is:

\[
\ddot{a}(t) = \frac{2}{1} \langle 4 \cos t, 4 \sin t, 0 \rangle = \langle 8 \cos t, 8 \sin t, 0 \rangle \text{ cm/s}^2.
\]

When we integrate the acceleration, we get the velocity:

\[
\vec{v}(t) = \langle 8 \sin t + c_1, -8 \cos t + c_2, c_3 \rangle \text{ cm/s}
\]

We know the initial velocity is \((0, -2, 1)\) cm/s. However, this doesn't automatically mean that \(c_1 = 0\), \(c_2 = -2\) and \(c_3 = 1\). \textit{This next step is the key difference between unit 8 and unit 9:} Using the formula above, at time \(t = 0\),

\[
\vec{v}(0) = \langle 8 \sin 0 + c_1, -8 \cos 0 + c_2, c_3 \rangle = \langle c_1, -8 + c_2, c_3 \rangle \text{ cm/s}
\]

So in order for the initial velocity to be \((0, -2, 1)\), \(c_2\) has to be 6 and not \(-2\). This gives us a velocity of

\[
\vec{v}(t) = \langle 8 \sin t, -8 \cos t + 6, 1 \rangle \text{ cm/s}
\]

When we integrate again, we get the particle's position:

\[
\vec{s}(t) = \langle -8 \cos t + d_1, -8 \sin t + 6t + d_2, t + d_3 \rangle \text{ cm}
\]

Again, we use this formula at time \(t = 0\):

\[
\vec{s}(0) = \langle -8 \cos 0 + d_1, -8 \sin 0 + 6(0) + d_2, 0 + d_3 \rangle = \langle -8 + d_1, d_2, d_3 \rangle
\]

Since the starting position was \((0, 0, 0)\), \(d_1\) has to equal 8. This makes the trajectory

\[
\vec{s}(t) = \langle -8 \cos t + 8, -8 \sin t + 6t, t \rangle \text{ cm}.
\]

Work with your group members to solve these problems. Remember to be careful when solving for the constants of integration.
Problem 1:

An experimental aircraft weighing 320 pounds is traveling 2000 ft above the ground at a velocity of 1200 ft/sec in the direction of the positive $x$-axis. The net force on the vehicle is given by $\vec{F}(t) = \{t^2, 10 \sin t, 500 e^{-t}\}$ lbs at time $t$ seconds.

G1. What's the initial velocity vector $\vec{v}_0$?

G2. We know that the weight of the aircraft is 320 pounds. What is the mass of the aircraft?

G3. Now that you know the mass, you should be able to find the acceleration of the aircraft. (Hint: $\vec{F} = m\vec{a}$.)

G4. By integrating, find the trajectory of the aircraft. Be careful about using the initial conditions to solve for the constants of integration.

G5. To the nearest ten feet, what is the height of the aircraft after 10 seconds?

Problem 2:

A motorboat is 76 feet from shore, accelerating towards the shore with an acceleration of magnitude $f(t) = 4t + 1$ feet/min$^2$ at time $t$ minutes.

If its initial speed is 600 ft/min and it is travelling at an angle of $30^\circ$ away from the (perfectly straight) shoreline, how far will it be from shore 6 minutes later?

*Hint: Since the boat is travelling on the surface of the water, consider this a problem in two dimensions, not three.*

Problem 3:

In this problem, we introduce the **unit tangent vector** and the **unit normal vector**, which are used in calculating the forces exerted on a moving body. Imagine that you're on a roller coaster, travelling down a steep hill and around a curve. You're experiencing two forces: One is gravity, which is pulling the coaster down the hill, and is responsible for the sensation of the wind whipping across your face as you careen down the hill. The other is centripetal force, which causes you to “lean into” the curve and slam sideways into the sidewall of the car. The unit tangent vector and the unit normal vector indicate the directions of these two forces.

**Problem 3**: A particle travels along the helical trajectory $\vec{s}(t) = (\cos t)\hat{i} + (\sin t)\hat{j} + (5t)\hat{k}$.

G6. Use the formula $\hat{T} = \frac{1}{|\vec{s}'(t)|}\vec{s}'(t)$ to find $\hat{T}$, the unit tangent vector.

G7. Use the formula $\hat{N} = \frac{1}{|\hat{T}'(t)|}\hat{T}'(t)$ to find $\hat{N}$, the unit normal vector. You will probably find this vector harder to simplify than $\hat{T}$ (why?)

G8. Find the values of $\hat{T}$ and $\hat{N}$ at $t = \pi$ seconds.

We'll use these two vectors again in Unit 10.