Math 1224

RECITATION 10: Components of Acceleration

Congratulations! You’ve come to the end of the course! Good luck on Test 3 and the Final Exam.

Introduction

The **unit tangent vector** \( \hat{T} \) to a trajectory at a point is a unit vector in the current direction of the trajectory. The **unit normal vector** \( \hat{N} \) to the same trajectory is a unit vector perpendicular to the trajectory, on the side pointing in towards the curve:

In Unit 9, we covered formulas for calculating these vectors:

\[
\hat{T}(t) = \frac{\vec{r}'(t)}{||\vec{r}'(t)||} \\
\hat{N}(t) = \frac{\hat{T}'(t)}{||\hat{T}'(t)||}
\]

However, it can often be very difficult to compute \( \hat{N} \) directly. One of the reasons is that calculating \( \hat{T}' \) often requires using the Quotient Rule, and we then have to scale the result of the differentiation to a unit vector.

The **tangential component of acceleration** \( a_T \) is the scalar projection of the acceleration vector onto the unit tangent vector; the **normal component of acceleration** \( a_N \) is the scalar projection of the acceleration vector onto the unit normal vector.

We can use these formulas to calculate these components:

\[
a_T = \vec{a} \cdot \hat{T} \\
a_N = \vec{a} \cdot \hat{N}
\]

But since \( \hat{N} \) is hard to calculate, we often prefer to compute \( a_N \) indirectly by using the fact that \( \hat{T} \) and \( \hat{N} \) are perpendicular—and using the Pythagorean Theorem instead. Even when we use this shortcut, we still have to use the first formula \( a_T = \vec{a} \cdot \hat{T} \) to find the tangential component.

**Example:** Let \( \vec{r}(t) = \langle 4t^2, -1, t^3 \rangle \). Find the components of acceleration \( a_T \) and \( a_N \) at time \( t = 2 \).

**Solution:** We first begin by finding the velocity and acceleration at time \( t = 2 \):

\[
\vec{v}(t) = \langle 8t, 0, 3t^2 \rangle \\
\vec{v}(2) = \langle 16, 0, 12 \rangle \\
\vec{a}(t) = \langle 8, 0, 6t \rangle \\
\vec{a}(2) = \langle 8, 0, 12 \rangle
\]

Since \( \hat{T} = \frac{\vec{r}'}{||\vec{r}'||} = \frac{\vec{v}}{||\vec{v}||} \), we have enough information to calculate the unit tangent vector:

\[
\hat{T}(2) = \frac{\langle 16, 0, 12 \rangle}{||\langle 16, 0, 12 \rangle||} = \frac{\langle 16, 0, 12 \rangle}{20} = \langle \frac{4}{5}, 0, \frac{3}{5} \rangle
\]
Now that we have $\ddot{a}(2)$ and $\dot{T}(2)$, we can find $a_T$ using their dot product:

\[
a_T = (8, 0, 12) \cdot \left( \begin{array}{c} 4 \\ 5 \\ 0 \\ 3 \\ 5 \\ \end{array} \right) \\
= \frac{32}{5} + 0 + \frac{36}{5} \\
= \frac{68}{5} = 13.6
\]

The hard way to compute $a_N$ is to calculate $\dot{N}$ first and then use the formula $a_N = \ddot{a} \cdot \dot{N}$. Instead, since we know that $\|\ddot{a}\| = \sqrt{208} \approx 14.4$ and $a_T = 13.6$ at time $t = 2$, we can use the fact that $a_T$ and $a_N$ always form two legs of a right triangle and solve for $a_N$ that way:

So $a_N = \sqrt{\left(\|\ddot{a}\|\right)^2 - (a_T)^2} = \sqrt{(\sqrt{208})^2 - 13.6^2} = 4.8$

Complete these three exercises with your group:

**Exercise 1: Straightforward!**

For the trajectory $\ddot{r}(t) = \{t, t^2, -3t\}$ find the components of acceleration at time $t = 5$. (In other words, find both $a_T$ and $a_N$.)

**Exercise 2: Slightly more involved**

An object with a mass of 12 kg follows the trajectory $\ddot{r}(t) = \{-2t^2, -2 \cos t, -2 \sin t\}$ (the position is measured in meters). Based on the example and Exercise 1, you should be able to find the components of acceleration. Instead, find the magnitude of the net force acting on the object in the tangential and normal directions.

**Exercise 3: More involved still!**

A projectile is fired at ground level with an initial velocity of $v_0$ and an angle of $45^\circ$. Find a formula for the curvature of its trajectory at the moment the projectile reaches its maximum height. (Curvature $\kappa$ is given by the formula $\kappa = \frac{a_N}{\|\dot{v}\|^2}$.)