I. Integration in Spherical Coordinates
Spherical coordinates locate points in space with two angles and one distance.

A. Definition
Spherical coordinates represent a point \( P \) in space by ordered triples \((\rho, \phi, \theta)\) in which
1. \( \rho \) is the distance from \( P \) to the origin. \( \rho \geq 0 \)
2. \( \phi \) is the angle \( \overrightarrow{OP} \) makes with the positive \( z \)-axis.
3. \( \theta \) is the angle from cylindrical coordinates.

B. Equations Relating Spherical Coordinates to Cartesian and Cylindrical Coordinates
\[
\begin{align*}
 r &= \rho \sin \phi, \quad x = r \cos \theta = \rho \sin \phi \cos \theta, \\
 z &= \rho \cos \phi, \quad y = r \sin \theta = \rho \sin \phi \sin \theta, \\
 \rho &= \sqrt{x^2 + y^2 + z^2} = \sqrt{r^2 + z^2}
\end{align*}
\]

C. Descriptions of \( \rho = a, \phi = \phi_0, \theta = \theta_0 \)
1. \( \rho = a \) describes a sphere of radius \( a \) centered at the origin.
2. \( \phi = \phi_0 \) describes a single cone whose vertex lies at the origin and whose axis lies along the \( z \)–axis.
   a. If \( \phi > \frac{\pi}{2} \) the cone \( \phi = \phi_0 \) opens downward
   b. If \( \phi < \frac{\pi}{2} \) the cone \( \phi = \phi_0 \) opens upward
   c. \( \phi = \frac{\pi}{2} \) is the \( xy \)-plane
   d. If \( \phi = 0 \) is the positive \( z \)-axis
   e. If \( \phi = \pi \) is the negative \( z \)-axis
3. \( \theta = \theta_0 \) describes a half-plane that contains the \( z \)–axis and makes an angle \( \theta_0 \) with the positive \( x \)-axis

D. Integrals in Spherical Form
1. Volume of a wedge: \( \Delta V = \Delta \rho \rho \Delta \phi \rho \sin \phi \Delta \theta = \rho^2 \sin \phi \Delta \rho \Delta \phi \Delta \theta \)
2. Volume: \( V = \iiint_D dV = \iiint_D \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta \)
3. \( \iiint_D f \, dV = \iiint_D f(\rho, \phi, \theta) \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta \)
E. Evaluating Triple Integral in which the Integrand is the Product of Three Functions of Different Variable with Constant Limits

**ONLY** if the integrand is the product of 3 functions of different variables and the limits of integration are constants: \[ \int_a^b \int_c^d \int_e^f g(x)h(y)k(z) \, dz \, dy \, dx = \int_a^b g(x)dx \int_c^d h(y)dy \int_e^f k(z)dz \]

F. Examples
1. Find the pt \( P = (x, y, z) \) whose spherical coordinates are \( \rho = 2, \phi = \frac{\pi}{3}, \theta = \frac{\pi}{4} \).

2. Find the spherical coordinates for the pt \( P = (0, 2\sqrt{3}, -2) \) given in rectangular coordinates.
3. Find the volume of the sphere of radius $a$, $x^2 + y^2 + z^2 = a^2$ using spherical coordinates.
2. Set up the integral in spherical form to find the volume of the “ice cream cone” cut from the solid sphere \( \rho \leq 5 \) by the cone \( \phi = \frac{\pi}{6} \).

3. Same region as in #2, but you “bite off the bottom of the cone” at \( \rho = 1 \).
4. Set up the integrals in spherical form to find the $z$-coordinate, $\bar{Z}$, of the center of mass of the top half of a hollow sphere whose outer radius is 4 and the inner radius is 3, if the density is $\delta(x, y, z) = x^2$.

5. Set up an iterated triple integral in spherical coordinates for the volume over the inside the sphere $x^2 + y^2 + (z - 2)^2 = 4$ and above the cone $z = \frac{1}{\sqrt{3}} \sqrt{x^2 + y^2}$.