I. Review: Center of Mass in 2-D

A. Definitions

1. Moment, \( M_0 \) = (mass)(directed distance)
2. Moment about the y-axis, \( M_y \) = (mass)(directed distance from the y-axis)
3. Moment about the x-axis, \( M_x \) = (mass)(directed distance from the x-axis)

B. Formulas for Moment, Mass & Center of Mass of a Lamina with Density Function, \( \delta \).

\[
\text{Mass: } M = \int_a^b dm = \int_a^b \delta \, dA
\]
\[
\text{Moment about the y-axis: } M_y = \int_a^b \tilde{x} \, dm = \int_a^b \tilde{x} \, \delta \, dA
\]
\[
\text{Moment about the x-axis: } M_x = \int_a^b \tilde{y} \, dm = \int_a^b \tilde{y} \, \delta \, dA
\]
\[
\text{Center of Mass: } \bar{X} = \frac{M_y}{M} , \quad \bar{Y} = \frac{M_x}{M}
\]

C. When integrating wrt \( x \):

\[
\tilde{x} = x , \quad \tilde{y} = \frac{t(x) + b(x)}{2} \quad \text{and} \quad dA = \left[ t(x) - b(x) \right] \, dx , \quad \text{where } t(x) = \text{top curve,}
\]
\[
b(x) = \text{bottom curve and } \delta(x) = \text{density function constant.}
\]

The formulas above can be rewritten as

\[
\text{Mass: } M = \int_a^b dm = \int_a^b \delta(x) \, \left( t(x) - b(x) \right) \, dx
\]
\[
\text{Moment about the y-axis: } M_y = \int_a^b \delta(x) \, x \, \left( t(x) - b(x) \right) \, dx
\]
\[
\text{Moment about the x-axis: } M_x = \int_a^b \delta(x) \, \left( \frac{t(x) + b(x)}{2} \right) \, \left( t(x) - b(x) \right) \, dx = \frac{1}{2} \int_a^b \delta(x) \, \left[ \left( t(x) \right)^2 - \left( b(x) \right)^2 \right] \, dx
\]
\[
\text{Center of Mass: } \bar{X} = \frac{M_y}{M} , \quad \bar{Y} = \frac{M_x}{M}
\]
D. When integrating wrt $y$:

$$\bar{x} = \frac{r(y) + l(y)}{2}, \quad \bar{y} = y \quad \text{and} \quad dA = [r(y) - l(y)] \, dy,$$

where $r(y)$ = right curve, $l(y)$ = left curve and $\delta(y)$ = density function.

The formulas above can be rewritten as

**Mass:**

$$M = \int_a^b dm = \int_a^b \delta(y) \left( r(y) - l(y) \right) dy$$

**Moment about the $x$-axis:**

$$M_x = \int_a^b [\delta(y)] \, y \left( r(y) - l(y) \right) dy$$

**Moment about the $y$-axis:**

$$M_y = \int_a^b \delta(y) \left( \frac{r(y) + l(y)}{2} \right) \left( r(y) - l(y) \right) dy$$

$$= \frac{1}{2} \int_a^b \delta(y) \left( \left[ r(y) \right]^2 - \left[ l(y) \right]^2 \right) dy$$

**Center of Mass:**

$$\bar{X} = \frac{M_y}{M}, \quad \bar{Y} = \frac{M_x}{M}$$

II. Center of Mass of Thin Plates using Double Integrals

A. Formulas

**Mass:**

$$M = \iint_R \delta(x,y) \, dA \quad \text{where} \quad \delta(x,y) \text{ is the density at } (x,y)$$

**First moments:**

$$M_y = \iint_R x \, \delta(x,y) \, dA$$

$$M_x = \iint_R y \, \delta(x,y) \, dA$$

**Center of Mass:**

$$\bar{X} = \frac{M_y}{M}, \quad \bar{Y} = \frac{M_x}{M}$$
B. Examples

1. Find the center of mass of a thin plate covering the region bounded by \( y = 1 - x^2 \) and \( y = x - 1 \), if the plates density at the point \((x, y)\) is \( \delta(x, y) = x^3 \).
2. Set up the integral formulas for the center of mass of a thin plate bounded by the $y$-axis and the lines $y = x$ and $y = 2 - x$ if the density is $\delta(x,y) = 6x + 3y + 3$. 