I. Types of Regions

We will be discussing two types of general regions in this section

A. Type I

An \(xy\)-region on the \(xy\)-plane is called a type I region if any vertical strip (in the \(y\) direction) always has the same upper and lower boundaries and the set can be described by the set of inequalities \(a \leq x \leq b, \ g_1(x) \leq y \leq g_2(x)\).

B. Type II

An \(xy\)-region on the \(xy\)-plane is called a type II region if any horizontal strip (in the \(x\) direction) always has the same right and left boundaries and the set can be described by the set of inequalities \(c \leq y \leq d, \ h_1(y) \leq x \leq h_2(y)\).

II. Fubini’s Theorem (Stronger Form)

Let \(f(x,y)\) be continuous on a region \(R\).

1. If \(R\) is defined by \(a \leq x \leq b, \ g_1(x) \leq y \leq g_2(x)\), with \(g_1(x)\) and \(g_2(x)\) continuous on \([a,b]\), then

\[
\iint_R f(x,y)dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)dydx.
\]

2. If \(R\) is defined by \(c \leq y \leq d, \ h_1(y) \leq x \leq h_2(y)\), with \(h_1(y)\) and \(h_2(y)\) continuous on \([c,d]\), then

\[
\iint_R f(x,y)dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)dxdy.
\]
III. Finding Limits of Integration

A. Integrating With Respect to \( y \) Then With Respect to \( x \)
   1. Sketch the region \( R \) (It should be Type I). Make sure to label the boundaries.
   2. Find the \( y \)-limits of integration. These are usually functions of \( x \). Lower limit=bottom curve, upper limit=top curve
   3. Find the \( x \)-limits of integration. These are usually constants. Lower limit=smallest \( x \)-value, upper limit=largest \( x \)-value

B. Integrating With Respect to \( x \) Then With Respect to \( y \)
   1. Sketch the region \( R \) (It should be Type II). Make sure to label the boundaries.
   2. Find the \( x \)-limits of integration. These are usually functions of \( y \). Lower limit=left curve, upper limit=right curve
   3. Find the \( y \)-limits of integration. These are usually constants. Lower limit=smallest \( y \)-value, upper limit=largest \( y \)-value

IV. Properties of Double Integrals

If \( f(x,y) \) and \( g(x,y) \) are continuous, then

1. Constant Multiple: \( \iint_{R} cf(x,y)\,dA = c \iint_{R} f(x,y)\,dA \) (any number \( c \))

2. Sum and Difference: \( \iint_{R} (f(x,y) \pm g(x,y))\,dA = \int \int_{R} f(x,y)\,dA \pm \int \int_{R} g(x,y)\,dA \)

3. Domination:
   a. \( \int \int_{R} f(x,y)\,dA \geq 0 \) if \( f(x,y) \geq 0 \) on \( R \)
   b. \( \int \int_{R} f(x,y)\,dA \geq \int \int_{R} g(x,y)\,dA \) if \( f(x,y) \geq g(x,y) \) on \( R \)

4. Additivity: \( \int \int_{R} f(x,y)\,dA = \int \int_{R_1} f(x,y)\,dA + \int \int_{R_2} f(x,y)\,dA \) if \( R \) is the union of two nonoverlapping regions \( R_1 \) and \( R_2 \)

5. Area of Bounded Regions in the Plane Using Double Integration
   The area of a closed, bounded plane region \( R \) is \( A = \int \int_{R} dA = \int_{a}^{b} \int_{h_1(x)}^{h_2(x)} dy\,dx = \int_{c}^{d} \int_{b_1(y)}^{b_2(y)} dx\,dy \)

6. If \( m \leq f(x,y) \leq M \forall (x,y) \) in \( D \), then \( mA \leq \int \int_{D} f(x,y)\,dA \leq MA \)
V. Examples

A. Sketch the region of integration and evaluate \[ \int_{1-x}^{2} \int_{x}^{2} (8x - 10y + 2) \, dy \, dx \]

B. \( \int_{x}^{y} \frac{y}{x} \, dA \) over the region R bounded by \( y = x \), \( y = 2x \), \( x = 1 \) and \( x = 2 \).
C. Write an equivalent double integral with the order of integration reversed and evaluate.
\[ \int_0^\pi \int_0^x \frac{\sin y}{y} \, dy \, dx \]

D. Find the area of the region bounded by \( y = x + 2 \) and \( y = x^2 \).
E. Find the volume of the region bounded by \( z = x^2 y \) and below by the region enclosed by \( y = x^2 \) and \( y = x \).

Write an equivalent double integral with the order of integration reversed.