I. Section 14.7 Review

Absolute Maxima and Minima on Closed Bounded Regions

To find absolute extrema for a continuous function $f(x,y)$ that is on a closed and bounded region $R$.

1. Find the critical points of the interior
2. List the intersection points of boundaries/endpoints/corners
3. Find the critical points of the boundaries
4. Evaluate $f$ at the points above
5. The largest value found is the absolute maximum of $f$ on $R$. The smallest value found is the absolute minimum of $f$ on $R$.

II. Lagrange Multipliers

A. Concept

You start at pt $Q$ and want to increase the value of $f$ while remaining on the level/constraint curve. You know that the gradient points in the direction of maximum increase, but you can’t move in this direction, because you would then move off of the constraint curve. Since the gradient $\nabla f_Q$ points to the right, you can increase $f$ somewhat by moving to the right along the constraint curve. Move to the right until you arrive at pt $P$, where the gradient $\nabla f_P$ is orthogonal to the constraint curve. Thus $f(P)$ is a local maximum subject to the constraint.

Vector $\nabla g_P$ is also orthogonal to the constraint curve, so $\nabla f_P$ and $\nabla g_P$ point in the same direction or in opposite directions. $\nabla f_P = \lambda \nabla g_P$ for some scalar $\lambda$ (Lagrange Multiplier).

Graphically, this means that a local maximum subject to the constraint occurs at pt $P$ where the level curves of $f$ and $g$ are tangent.
B. Method Of Lagrange Multipliers

To find the maximum and minimum values of \( f(x,y,z) \) subject to the constraint \( g(x,y,z) = k \) (assuming that these extreme values exist and \( \nabla g \neq 0 \) on the surface \( g(x,y,z) = k \)):

a) Find all values of \( x, y, z \) and \( \lambda \) such that \( \nabla f(x,y,z) = \lambda \nabla g(x,y,z) \)
(i.e., \( f_x = \lambda g_x \) & \( f_y = \lambda g_y \)) and \( g(x,y,z) = k \)

b) Evaluate \( f \) at all the points \((x,y,z)\) that result from step (a). The largest of these values is the maximum value of \( f \); the smallest is the minimum value of \( f \).

C. Examples

1. Find the extrema of the function \( f(x,y) = xy \) on the ellipse \( \frac{x^2}{8} + \frac{y^2}{2} = 1 \).

   When subjected to the constraint \( g(x,y) = \frac{x^2}{8} + \frac{y^2}{2} - 1 = 0 \), the function \( f(x,y) = xy \) takes the extreme values at the points on the ellipse where \( \nabla f \) (red) is a scalar multiple of \( \nabla g \) (blue).
2. Find the extrema of the function \( f(x,y) = x^2 + 2y^2 \) on the disk \( x^2 + y^2 \leq 1 \).

The extrema of \( f(x,y) = x^2 + 2y^2 \) correspond to either the points where the level curves touch the circle \( x^2 + y^2 = 1 \) or at the critical point of \( f \).
3. Find the points on the sphere \( x^2 + y^2 + z^2 = 4 \) that are closest to farthest from the point \((3,1,-1)\).
D. Two Constraint Method Of Lagrange Multipliers

To find the maximum and minimum values of \( f(x,y,z) \) subject to two constraints \( g(x,y,z)=k \) and \( h(x,y,z)=c \) look for extrema of \( f \) where \((x,y,z)\) is restricted to lie on the curve of the intersection of \( C \) of the level surfaces \( g(x,y,z)=k \) and \( h(x,y,z)=c \).

Find all values of \( x, y, z, \lambda \) and \( \mu \) such that \( \nabla f(x,y,z) = \lambda \nabla g(x,y,z) + \mu \nabla h(x,y,z) \) and \( g(x,y,z)=k \) and \( h(x,y,z)=c \).

Solve the five equations in five unknowns:
\[
\begin{align*}
  f_x &= \lambda g_x + \mu h_x, \\
  f_y &= \lambda g_y + \mu h_y, \\
  f_z &= \lambda g_z + \mu h_z, \\
  g(x,y,z) &= k, \\
  h(x,y,z) &= c.
\end{align*}
\]

E. Example

Find the extrema of the function \( f(x,y,z) = x + 2y \) on the curve of intersection of the plane \( x + y + z = 1 \) and the cylinder \( y^2 + z^2 = 4 \).