Math 2204 Multivariable Calculus – Chapter 14: Partial Derivatives
Sec. 14.7: Maximum and Minimum Values

I. Review from 1225
A. Definitions
1. Local Extreme Values (Relative)
   a. A function \( f \) has a local maximum value at an interior point \( c \) of its domain if \( f(x) \leq f(c) \) for all \( x \) in an open interval containing \( c \).
   b. A function \( f \) has a local minimum value at an interior point \( c \) of its domain if \( f(x) \geq f(c) \) for all \( x \) in an open interval containing \( c \).

2. Critical Number
   a. A critical number of a function \( f \) is a number \( c \) in the domain of \( f \) such that \( f'(c) \) is zero or \( f'(c) \) does not exist (undefined). (A stationary point exists where \( f'(x) = 0 \) and a singular point exists where \( f'(x) \) is undefined.)
   b. If \( f \) has a local maximum or minimum at \( c \), then \( c \) is a critical number of \( f \).

3. A point where the graph of a function has a tangent line and where the concavity changes is called a point of inflection.

B. The First Derivative Test for Local Extrema
   Let \( f \) be a continuous function on \([a,b]\) and \( c \) be a critical number in \([a,b]\).
   1. If \( f'(x) \geq 0 \) on \((a,c)\) and \( f'(x) \leq 0 \) on \((c,b)\), then \( f \) has a local maximum of \( f(c) \) at \( x = c \).
   2. If \( f'(x) \leq 0 \) on \((a,c)\) and \( f'(x) \geq 0 \) on \((c,b)\), then \( f \) has a local minimum of \( f(c) \) at \( x = c \).
   3. If \( f'(x) \) does not change signs at \( x = c \), then \( f \) has no local extrema at \( x = c \).

C. The Second Derivative Test for Local Extrema
   Let \( f \) be a continuous function on \([a,b]\) and \( c \) be a critical point in \([a,b]\).
   1. If \( f''(c) < 0 \), then \( f \) has a local maximum of \( f(c) \) at \( x = c \).
   2. If \( f''(c) > 0 \), then \( f \) has a local minimum of \( f(c) \) at \( x = c \).
   3. If \( f''(c) = 0 \), then you must go back and use the First Derivative Test for Local Extrema

D. The to find absolute extrema for a function \( f \) that is continuous on a closed interval \([a,b]\).
   1. Find all critical numbers for \( f \) in \((a,b)\). (i.e. \( f'(x) \) is zero or undefined)
   2. Evaluate \( f \) at all critical numbers in \((a,b)\).
   3. Evaluate \( f \) at the endpoints \( a \) and \( b \) of the interval \([a,b] \).
4. The largest value found in Steps 2 and 3 is the absolute maximum for $f$ on $[a,b]$. The smallest value found in Steps 2 and 3 is the absolute minimum for $f$ on $[a,b]$.

II. Derivative Test for Local Extreme Values

A. Definitions

1. An interior point of the domain of a function $f(x,y)$ where both $f_x$ and $f_y$ are zero or where one or both of $f_x$ and $f_y$ do not exist is a critical point of $f$.

2. Let $f(x,y)$ be defined on a region $R$ containing the point $(a,b)$. Then
   a. $f(a,b)$ is a local maximum value of $f$ if $f(a,b) \geq f(x,y)$ for all domain points $(x,y)$ in an open disk centered at $(a,b)$.
   b. $f(a,b)$ is a local minimum value of $f$ if $f(a,b) \leq f(x,y)$ for all domain points $(x,y)$ in an open disk centered at $(a,b)$.

3. A differentiable function has a saddle point at a critical point $(a,b)$ if in every open disk centered at $(a,b)$ there are domain points $(x,y)$ where $f(x,y) > f(a,b)$ and domain points $(x,y)$ where $f(x,y) < f(a,b)$. The corresponding point $(a,b,f(a,b))$ on the surface $z = f(x,y)$ is called a saddle point of the surface.

B. Theorems

1. Theorem 10: First Derivative Test for Local Extreme Values

   If $f(x,y)$ has a local maximum or minimum value at an interior point $(a,b)$ of its domain and if the first partial derivatives exist there, then $f_x(a,b) = 0$ and $f_y(a,b) = 0$.

2. Theorem 11: Second Derivative Test for Local Extreme Values

   Suppose that $f(x,y)$ and its first and second partial derivatives are continuous throughout a disk centered $(a,b)$ at and that $f_x(a,b) = f_y(a,b) = 0$. Then
   i. $f$ has a local maximum at $(a,b)$ if $f_{xx} < 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$ at $(a,b)$.
   ii. $f$ has a local minimum at $(a,b)$ if $f_{xx} > 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$ at $(a,b)$.
   iii. $f$ has a saddle point at $(a,b)$ if $f_{xx}f_{yy} - f_{xy}^2 < 0$ at $(a,b)$.
   iv. The test is inconclusive at $(a,b)$ if $f_{xx}f_{yy} - f_{xy}^2 = 0$ at $(a,b)$. In this case, we must find some other way to determine the behavior of $f$ at $(a,b)$. 

\[
\begin{align*}
z &= -x^2 - y^2 \\
z &= x^2 + y^2 \\
z &= -x^2 + y^2
\end{align*}
\]

Local max 
Local min 
Saddle point
Note: $f_{xx}f_{yy} - f_{xy}^2$ is called the discriminant or Hessian of $f$.

C. Examples

1. Find the local extrema of the function $f(x, y) = x^4 + y^3 + 32x - 9y$.

2. Find the local extrema of the function $f(x, y) = xy^2 - 6x^2 - 3y^2 + 4$. 
3. Find the local extrema of the function \( f(x,y) = 6x^2 - 2x^3 + 3y^2 + 6xy - 3 \).

II. Absolute Maxima and Minima on Closed Bounded Regions
   A. Closed vs Open, Bounded vs Nonbounded
      
      B. Absolute extrema for a continuous function \( f(x,y) \) that is defined on a closed and bounded region \( R \) will always be at a critical points or on the boundary

C. To find absolute extrema for a continuous function \( f(x,y) \) that is on a closed and bounded region \( R \).
   1. Find the critical points of the interior
   2. List the intersection points of bounderies/endpoints/corners
   3. Find the critical points of the boundaries
   4. Evaluate \( f \) at the points above
5. The largest value found is the absolute maximum of $f$ on $R$. The smallest value found is the absolute minimum of $f$ on $R$.

D. Examples
1. Find the absolute extrema of $f(x,y) = x^3 + y^4 - 12x + 4y$ on the rectangular plate $-3 \leq x \leq 0, -2 \leq y \leq 0$. 

![Diagram of a rectangular plate with coordinates x from -3 to 0 and y from -2 to 0]
2. Find the absolute extrema of \( f(x, y) = x^3 + y^4 - 12x + 4y \) on the triangular plate bounded by the lines \( x = 2, y = 0, y = 2x \).
3. Find the absolute extrema of $f(x,y) = x + y^2$ on the circular region $x^2 + y^2 \leq 1$. 