Math 2204 Multivariable Calculus – Chapter 13: Vector Functions
Sec. 13.2: Derivatives and Integrals of Vector Functions

I. Derivatives of Vector Functions

A. Definitions

1. Derivative
\[ \frac{d\vec{r}}{dt} = \vec{r}'(t) = \lim_{h \to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}, \] provided the limit exists.

2. The vector \( \vec{r}'(t) \) is called the tangent vector to the curve defined by \( \vec{r}(t) \) at the point \( P \), provided that \( \vec{r}'(t) \) exists and \( \vec{r}'(t) \neq 0 \). The tangent line to \( C \) at \( P \) is defined to be the line through \( P \) parallel to the tangent vector \( \vec{r}'(t) \).

3. The unit tangent vector is \( \vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|} \).

B. Theorem

If \( \vec{r}(t) = \langle f(t), g(t), h(t) \rangle = f(t) \vec{i} + g(t) \vec{j} + h(t) \vec{k} \), where \( f, g \) and \( h \) are differentiable functions, then
\[ \vec{r}(t) = \langle f'(t), g'(t), h'(t) \rangle = f'(t) \vec{i} + g'(t) \vec{j} + h'(t) \vec{k}. \]

C. Differentiation Rules

Theorem: Suppose \( \vec{u} \) and \( \vec{v} \) are differentiable vector functions, \( c \) is a scalar, and \( f \) is a real-valued function. Then

1. \[ \frac{d}{dt} [\vec{u}(t) + \vec{v}(t)] = \vec{u}'(t) + \vec{v}'(t) \]
2. \[ \frac{d}{dt} [c \vec{u}(t)] = c \vec{u}'(t) \]
3. \[ \frac{d}{dt} [f(t) \vec{u}(t)] = f'(t) \vec{u}(t) + f(t) \vec{u}'(t) \]
4. \[ \frac{d}{dt} [\vec{u}(t) \cdot \vec{v}(t)] = \vec{u}'(t) \cdot \vec{v}(t) + \vec{u}(t) \cdot \vec{v}'(t) \]
5. \[ \frac{d}{dt} [\vec{u}(t) \times \vec{v}(t)] = \vec{u}'(t) \times \vec{v}(t) + \vec{u}(t) \times \vec{v}'(t) \]
6. \[ \frac{d}{dt} [\vec{u}(f(t))] = f'(t)\vec{u}'(f(t)) \] (Chain Rule)
D. If $|\vec{r}(t)| = c$, then $\vec{r}'(t)$ is orthogonal to $\vec{r}(t)$ for all $t$.

This can be shown using differentiation rule 4 above. Geometrically, if a curve lies on a sphere with the center at the origin, the tangent vector $\vec{r}'(t)$ is always perpendicular to the position vector $\vec{r}(t)$.

E. Examples

1. Given $\vec{r}(t) = \langle te^{-t}, 2\tan^{-1} t, 2e^t \rangle$
   
a. Find the derivative of $\vec{r}(t) = \langle te^{-t}, 2\tan^{-1} t, 2e^t \rangle$.

   b. Find the unit tangent vector at the point where $t = 0$.

2. For the curve $\vec{r}(t) = (1 + \cos(t)) \vec{i} + (2 + \sin(t)) \vec{j}$, find $\vec{r}'(t)$ and sketch the position vector $\vec{r}\left(\frac{\pi}{6}\right)$ and the tangent vector $\vec{r}'\left(\frac{\pi}{6}\right)$.
3. Find parametric equations for the tangent line to the curve with the parametric equation 
\[ x = \ln(t + 1), \ y = t \cos(2t), \ z = t \] at the point \( (0,0,1) \).

4. If \( \vec{r}(t) = \left\langle e^{2t}, e^{-2t}, te^{-t} \right\rangle \), find \( \vec{r}'(t), \vec{r}''(t), \vec{T}(1), \vec{r}''(t), \) and \( \vec{r}'(t) \cdot \vec{r}''(t) \).
II. Integrals

A. Definite Integral

1. Definition

\[
\int_a^b \bar{r}(t) \, dt = \lim_{n \to \infty} \sum_{i=1}^{n} \bar{r}(t_i^*) \Delta t = \lim_{n \to \infty} \left[ \left( \sum_{i=1}^{n} f(t_i^*) \Delta t \right) \bar{i} + \left( \sum_{i=1}^{n} g(t_i^*) \Delta t \right) \bar{j} + \left( \sum_{i=1}^{n} h(t_i^*) \Delta t \right) \bar{k} \right]
\]

\[
\Rightarrow \int_a^b \bar{r}(t) \, dt = \left( \int_a^b f(t) \, dt \right) \bar{i} + \left( \int_a^b g(t) \, dt \right) \bar{j} + \left( \int_a^b h(t) \, dt \right) \bar{k}
\]

2. Examples

a. Evaluate \[
\int \left( te^{2t} \bar{i} + \frac{t}{1-t} \bar{j} + \frac{1}{\sqrt{1-3t^2}} \bar{k} \right) \, dt
\]
b. Find \( \vec{r}(t) \) if \( \vec{r}'(t) = \langle 2t, 3t^2, \sqrt{t} \rangle \) and \( \vec{r}(1) = \langle 1, 1, 0 \rangle \)