Math 2204 Multivariable Calculus – Chapter 12: Vectors and the Geometry of Space
Sec. 12.3: The Dot Product

I. Dot Product

A. Definition

If \( \vec{a} = <a_1, a_2, a_3> \) and \( \vec{b} = <b_1, b_2, b_3> \), then the dot product of \( \vec{a} \) and \( \vec{b} \) is the number \( \vec{a} \cdot \vec{b} \) given by
\[
\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2 + a_3b_3
\]

B. Properties of the Dot Product

If \( \vec{a}, \vec{b}, \) and \( \vec{c} \) are vectors in \( V_3 \) and \( c \) is a scalar, then

1. \( \vec{a} \cdot \vec{a} = |\vec{a}|^2 \)
2. \( \vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a} \)
3. \( \vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c} \)
4. \( (c\vec{a}) \cdot \vec{b} = c(\vec{a} \cdot \vec{b}) = \vec{a} \cdot (c\vec{b}) \)
5. \( \vec{0} \cdot \vec{a} = 0 \)

C. Theorem

If \( \theta \) is the angle between the vectors \( \vec{a} \) and \( \vec{b} \), then
\[
\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos \theta
\]

Corollary: If \( \theta \) is the angle between the nonzero vectors \( \vec{a} \) and \( \vec{b} \), then
\[
\cos \theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}
\]

D. Dot Product and Angles

1. If the dot product of two vectors is positive, the vectors form an acute angle.
2. If the dot product of two vectors is negative, the vectors form an obtuse angle.
3. If the dot product of two vectors is zero, the vectors form a right angle.

*Orthogonal Vectors: Two vectors \( \vec{a} \) and \( \vec{b} \) are orthogonal (perpendicular, normal) iff \( \vec{a} \cdot \vec{b} = 0 \).

Proof:

E. Examples

1. Find \( <3,2> \cdot <4,1> \).
2. Find the angle between $<-9,9,1>$ and $<4,-4,8>$ and then approximate the angle to the nearest degree.

3. Determine whether the given vectors are orthogonal, parallel or neither.
   (Two vectors $\vec{a}$ and $\vec{b}$ are parallel if $\vec{a} = t\vec{b}$ for some number $t$)
   a. $<-1,2,5> \& <3,4,-1>

   b. $\vec{a} = 2\vec{i} + 6\vec{j} - 4\vec{k}$ \& $\vec{b} = -3\vec{i} - 9\vec{j} + 6\vec{k}$

   c. $<1,-1,2> \& <2,-1,1>$
II. Projections
A. Definitions
1. Let $P$, $Q$ and $R$ be points and suppose $\vec{a} = \overrightarrow{PQ}$ and $\vec{b} = \overrightarrow{PR}$. Let $L$ be the line containing points $P$ and $Q$ and let $S$ be the orthogonal projection of $R$ onto $L$. The vector $\overrightarrow{PS}$ is called the vector projection of $\vec{b}$ onto $\vec{a}$ and is denoted $\text{proj}_\vec{a}\vec{b}$.

2. The scalar projection of $\vec{b}$ onto $\vec{a}$ (also called the component of $\vec{b}$ onto $\vec{a}$) is the signed magnitude of the vector projection of $\vec{b}$ onto $\vec{a}$ and is denoted $\text{comp}_\vec{a}\vec{b}$.

B. Examples
1. Find the vector projection of $\vec{b} = \vec{i}$ onto $\vec{a} = \vec{i} + \vec{j}$.

2. Find the scalar projection of $\vec{b} = \vec{i}$ onto $\vec{a} = -\vec{i} + \vec{j}$.

3. Let $\theta$ be the angle between $\vec{a}$ and $\vec{b}$. What can be said about $\text{comp}_\vec{a}\vec{b}$ if $\theta = \frac{\pi}{2}$, $\theta < \frac{\pi}{2}$, $\theta > \frac{\pi}{2}$?
C. Formulas

1. Scalar projection of $\vec{b}$ onto $\vec{a}$:  
   $$\text{comp}_a \vec{b} = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|}$$

2. Vector projection of $\vec{b}$ onto $\vec{a}$:  
   $$\text{proj}_a \vec{b} = \left( \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|^2} \right) \vec{a}$$

   We can write the vector projection of $\vec{b}$ onto $\vec{a}$ as $\text{proj}_a \vec{b} = s \frac{\vec{a}}{|\vec{a}|}$ where $s$ is the scalar projection of $\vec{b}$ onto $\vec{a}$.

D. Examples

1. Find the scalar projection and vector projection of $\vec{b} = \vec{i} + 2\vec{j} - 4\vec{k}$ onto $\vec{a} = 2\vec{i} + \vec{j} + 2\vec{k}$.

2. Suppose a force $\vec{F}$ moves an object from point $P$ to point $Q$ and let $\vec{D} = \vec{PQ}$. The work done by the force is defined to be the product of the component of the force along $\vec{D}$ and the distance moved. Show the work done by the force is given by $W = \vec{F} \cdot \vec{D}$. 
3. Find the work done by a force $\vec{F} = 8\vec{i} - 6\vec{j} + 9\vec{k}$ that moves an object from pt $P (0,10,8)$ to pt $Q (6,12,20)$ along a straight line. The distance is measured in meters and the force in newtons.

4. A sled is pulled along a level path through snow by a rope. A 30-lb force acting at an angle of $40^\circ$ above the horizontal moves the sled 80 ft. Find the work done by the force.
5. Use a scalar projection to show that the distance from a pt \( P_0 = (x_0, y_0) \) to the line with an equation \( ax + by + c = 0 \) is given by \[ d = \frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}} \].

Use this formula to find the distance from the pt \((-2, 3)\) to the line \(3x - 4y + 5 = 0\)