Math 1206 Calculus - Sec. 4.4: Estimating with Finite Sums

I. Consider the problem of finding the area under the curve on the fn $y = x^2 + 5$ over the domain $[0, 2]$. We can approximate this area by using a familiar shape, the rectangle. If we divide the domain interval into several pieces, then draw rectangles having the width of the pieces, and the height of the curve, we can get a rough idea of the total area.

For example suppose we divide the interval $[0, 2]$ into 5 subintervals each of width $2 / 5$.
$[0, .4], [.4, .8], [.8, 1.2], [1.2, 1.6], [1.6, 2.0]$.

The table below shows the values obtained when $y(x)$ is evaluated at the corresponding points.

<table>
<thead>
<tr>
<th>time(sec)</th>
<th>velocity(ft/sec)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>5</td>
</tr>
<tr>
<td>0.4</td>
<td>4.84</td>
</tr>
<tr>
<td>0.8</td>
<td>4.36</td>
</tr>
<tr>
<td>1.2</td>
<td>3.56</td>
</tr>
<tr>
<td>1.6</td>
<td>2.44</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
</tr>
</tbody>
</table>

Plotting these points yields the following graph.

If we find the minimum value in the subinterval, and use this as our height for that rectangle, we have what is known as an **inscribed rectangle**. See the graph below.

Now each of the above rectangles has the exact same width, namely $2 / 5$. For this function the height of each rectangle is given by calculating the value of the function at the right hand endpoint of each subinterval. The area under the curve, can then be approximated by adding the areas of all the rectangles together.

Notice that when using the minimum values, i.e. using inscribed rectangles, we arrive at an estimate that is lower than the actual area under the curve. Hence, this method results in what is known as the **lower sum**.

Let's calculate the above estimate: i.e. the Lower Sum.
Lower Sum = $\frac{2}{5} f\left(\frac{2}{5}\right) + \frac{2}{5} f\left(\frac{4}{5}\right) + \frac{2}{5} f\left(\frac{6}{5}\right) + \frac{2}{5} f\left(\frac{8}{5}\right) + \frac{2}{5} f(2)$

$= \frac{2}{5} \left[ f\left(\frac{2}{5}\right) + f\left(\frac{4}{5}\right) + f\left(\frac{6}{5}\right) + f\left(\frac{8}{5}\right) + f(2) \right]$

$= \frac{2}{5} [4.84 + 4.36 + 3.56 + 2.44 + 1]$

$= \frac{2}{5} [16.2]$

$= 6.48$

You can also calculate an estimate using the maximum value in the subinterval and using it as the height of the rectangles. These rectangles are known as **circumscribed rectangles**. The resulting area approximation will be greater than the area under the curve. Consequently, we call this type of sum an **upper sum**.

\[ v(t) = -t^2 + 5 \]

Let's calculate the above estimate; i.e. the Upper Sum

Upper Sum = $\frac{2}{5} f(0) + \frac{2}{5} f\left(\frac{2}{5}\right) + \frac{2}{5} f\left(\frac{4}{5}\right) + \frac{2}{5} f\left(\frac{6}{5}\right) + \frac{2}{5} f\left(\frac{8}{5}\right)$

$= \frac{2}{5} \left[ f(0) + f\left(\frac{2}{5}\right) + f\left(\frac{4}{5}\right) + f\left(\frac{6}{5}\right) + f\left(\frac{8}{5}\right) \right]$

$= \frac{2}{5} [5 + 4.84 + 4.36 + 3.56 + 2.44]$

$= \frac{2}{5} [20.2]$

$= 8.08$

From the two calculations above we can conclude that the area of the curve lies somewhere between the two approximations, i.e. $6.48 < \text{area of region} < 8.08$

Another method that can yield a better approximation is known as the **midpoint rule**. In the midpoint rule, you choose the value exactly in the middle of the subinterval to use in calculating the height of the rectangle; resulting in some rectangles being both inscribed & circumscribed.
Let's calculate the above estimate: i.e. the Average or Midpoint Sum.

\[
\text{Average Sum} = \frac{2}{5} f\left(\frac{1}{5}\right) + \frac{2}{5} f\left(\frac{3}{5}\right) + \frac{2}{5} f\left(\frac{1}{5}\right) + \frac{2}{5} f\left(\frac{7}{5}\right) + \frac{2}{5} f\left(\frac{9}{5}\right)
\]

\[
= \frac{2}{5} \left[ f\left(\frac{1}{5}\right) + f\left(\frac{3}{5}\right) + f\left(\frac{1}{5}\right) + f\left(\frac{7}{5}\right) + f\left(\frac{9}{5}\right) \right]
\]

\[
= \frac{2}{5} \left[ 4.96 + 4.64 + 4 + 3.04 + 1.76 \right]
\]

\[
= \frac{2}{5} [18.4]
\]

\[
= 7.36
\]

II. Things to note:

A. The smaller the subintervals, the better the approximation will be. This is because, the function's values are changing less in the subinterval, i.e. the value of the function is fairly constant in each subinterval. Consequently, we are not approximating by such a rough amount each time. For example, here is the same region divided into 20 rectangle instead of 5. Note that the error is minute compared with the previous work.

1. Each of the above processes (lower sum, upper sum, midpoint sum) are just approximations. They are not exact.

2. When you want to calculate the Volume of a solid, you can use similar techniques, only you'll be using rectangular solids or cylinders to approximate the volume.
III. Accuracy

Error Magnitude = |true value - calculated value|

Error = \[ \frac{\text{true value} - \text{calculated sum}}{\text{true value}} \]

Percentage Error = \[ \frac{\text{true value} - \text{calculated sum}}{\text{true value}} \times 100\% \]

For the example in part I, the true value for the area under the curve \( y = -x^2 + 5 \) over the domain \([0, 2]\) is \( \frac{22}{3} = 7.33 \). Therefore the error associated with the approximations are:

\[
\text{Error}_{\text{lower sum}} = \frac{\left| \frac{22}{3} - 6.48 \right|}{\frac{22}{3}} \approx 0.11636364
\]

\[
\text{Error}_{\text{upper sum}} = \frac{\left| \frac{22}{3} - 8.08 \right|}{\frac{22}{3}} \approx 0.1018182
\]

\[
\text{Error}_{\text{midpoint sum}} = \frac{\left| \frac{22}{3} - 7.36 \right|}{\frac{22}{3}} \approx 0.0036364
\]

IV. Average Value of a Non-negative Function

(a.) Formula:

\[ av(f) = \frac{1}{\text{length of interval } [a,b]} \left( \text{sum of function values using the midpoint sum} \right) \]

\[ av(f) = \frac{1}{\text{length of interval } [a,b]} (\text{Midpoint Sum}) \]

(b.) Example

Find the average value of \( y = x + 1 \) on \([1,2]\) using 4 subintervals.

\[ av(f) = \frac{1}{2 - 1} \left( f\left( \frac{9}{8} \right) + f\left( \frac{11}{8} \right) + f\left( \frac{13}{8} \right) + f\left( \frac{15}{8} \right) \right) \]

\[ av(f) = (1) \left( \frac{17}{8} + \frac{19}{8} + \frac{21}{8} + \frac{23}{8} \right) \left( \frac{1}{4} \right) \]

\[ av(f) = 2.5 \]

Find where the fn \( y = x + 1 \) on \([1,2]\) using 4 subintervals obtains the average value.

Set fn = av(f).

\( x + 1 = 2.5 \Rightarrow x = 1.5 \)
IV. More Examples
1. Oil is leaking out of a tanker damaged at sea. The damage to the tanker is worsening as evidenced by the increased leakage each hour.

<table>
<thead>
<tr>
<th>time (hours)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>leakage (gallons/hr)</td>
<td>50</td>
<td>70</td>
<td>97</td>
<td>136</td>
<td>190</td>
<td>265</td>
<td>369</td>
<td>516</td>
<td>720</td>
</tr>
</tbody>
</table>

Estimate the total amount of oil (in gallons) that leaks out during the first eight hours.

Using left hand endpoints

Using right hand endpoints

Using midpoints
2. The table below gives dye concentrations for a cardiac output determination. The amount of dye injected was 10mg.

<table>
<thead>
<tr>
<th>seconds after injection</th>
<th>0</th>
<th>4</th>
<th>8</th>
<th>12</th>
<th>16</th>
<th>20</th>
<th>24</th>
<th>28</th>
<th>32</th>
</tr>
</thead>
<tbody>
<tr>
<td>dye concentration (mg)</td>
<td>0</td>
<td>0.1</td>
<td>2</td>
<td>6.3</td>
<td>7.9</td>
<td>6.1</td>
<td>3.5</td>
<td>0.7</td>
<td>0</td>
</tr>
</tbody>
</table>

a. Plot the data and connect the data points with a smooth curve.

b. Estimate the area under the curve using midpoints. (units for area: \( \frac{mg}{L} \cdot \text{sec} \))

c. Calculate the cardiac output from this estimate.

\[
\text{cardiac output} = \left( \frac{\text{amount of dye}}{\text{area under curve}} \right) \cdot \frac{60 \text{sec}}{\text{min}} \rightarrow \text{units for output: } \left( \frac{L}{\text{min}} \right)
\]