I. Increasing/Decreasing Functions

A. Definition: Let \( f \) be a function defined on an interval \( I \) and let \( x_1 \) and \( x_2 \) be any 2 points in \( I \).

1. \( f \) increases on \( I \) if \( f(x_1) < f(x_2) \) whenever \( x_1 < x_2 \).
2. \( f \) decreases on \( I \) if \( f(x_1) > f(x_2) \) whenever \( x_1 < x_2 \).

A function that is increasing or decreasing on \( I \) is called monotonic on \( I \).

B. The First Derivative Test for Increasing/Decreasing.
Suppose that \( f \) is continuous on \([a, b]\) and differentiable on the open interval \((a, b)\).

1. A function, \( f \), increases on \((a, b)\) if \( f'(x) > 0 \) for all \( x \) in \((a, b)\).
2. A function, \( f \), decreases on \((a, b)\) if \( f'(x) < 0 \) for all \( x \) in \((a, b)\).
3. A function, \( f \), is constant on \((a, b)\) if \( f'(x) = 0 \) for all \( x \) in \((a, b)\).

C. Example:

1. On what intervals(s) is the graph of the function above:
   a. increasing: ________________________________
   b. decreasing: ________________________________
   c. constant: ________________________________

2. On what intervals(s) is the derivative, \( f'(x) \):
   a. positive: ________________________________
   b. negative: ________________________________
   c. zero: ________________________________
   d. undefined: ________________________________
II. Critical Value/Critical Point

A. Definitions

1. A critical value of a function $f$ is a number $c$ in the domain of $f$ such that $f'(c)$ is zero or $f'(c)$ does not exist (undefined).

2. A stationary point exists where $f''(x) = 0$.

3. A singular point exists where $f'(x)$ is undefined.

B. If $f$ has a relative maximum or minimum at $c$, then $c$ is a critical value of $f$.

C. Examples: Find the critical value(s) of the following:

1. $y = -2x^2 - 5x + 3$

2. $f(x) = x^3 + 3x^2 - 45x - 6$

3. $g(x) = \sqrt{9x - 4}$

III. Absolute Extreme Values / Absolute Extrema

A. Definitions: Let $f$ be a function with domain $D$.

1. $f$ has an absolute maximum value on $D$ at a point $c$ if: $f(x) \leq f(c)$ for all $x$ in $D$. The number $f(c)$ is called the maximum value of $f$ on $D$.

2. $f$ has an absolute minimum value on $D$ at a point $c$ if: $f(x) \geq f(c)$ for all $x$ in $D$. The number $f(c)$ is called the minimum value of $f$ on $D$.

3. The maximum and minimum values of $f$ are called the extreme values of $f$. 
B. Examples: Do the following functions have absolute extrema on the given interval?

1. \([-5,10]\]

   ![Graph of a function with absolute maximum and minimum marked]

   Absolute maximum:___________

   Absolute minimum:___________

2. \([1,11]\) Are you guaranteed to have absolute extrema?

   ![Graph of a function with absolute maximum and minimum marked]

   Absolute maximum:___________

   Absolute minimum:___________

3. \(f(x) = x^2 - 2x\) on \([-1,3]\)

   ![Graph of a function with absolute maximum and minimum marked]

   Absolute maximum:___________

   Absolute minimum:___________
4. \( f(x) = x^2 - 2x \) on \((-1, 2)\)

Absolute maximum: 

Absolute minimum: 

5. \( f(x) = x^2 - 2x \) on \([-1, 1), (1, 2]\)

Absolute maximum: 

Absolute minimum: 

C. Extreme Value Theorem

1. **Theorem 1** - The Extreme Value Theorem: If \( f \) is continuous at every point of a closed interval \([a, b]\), then \( f \) attains both an absolute maximum value \( M \) and an absolute minimum value \( m \) somewhere in \([a, b]\). That is, there are numbers \( x_1 \) and \( x_2 \) in \([a, b]\), with \( f(x_1) = m, \ f(x_2) = M \), and \( m \leq f(x) \leq M \) for every other \( x \) in \([a, b]\).

2. **Note**: If the hypotheses of this theorem are met, then you are GUARANTEED the existence of a Max or Min value. If the hypotheses are not met, you MAY have a max or min value.

D. Procedure To Find Absolute Extrema on a Closed Interval, \([a, b]\):

1. Verify that the function \( f \) is continuous on \([a, b]\).
2. Find all critical values for \( f \) in \((a, b)\). (i.e. when \( f'(x) = 0 \) or undefined)
3. Evaluate \( f \) at all critical numbers in \((a, b)\).
4. Evaluate \( f \) at the endpoints \( a \) and \( b \) of the interval \([a, b]\).
5. The largest value found in Steps 3 and 4 is the absolute maximum for \( f \) on \([a, b]\).
   The smallest value found in Steps 3 and 4 is the absolute minimum for \( f \) on \([a, b]\).
E. Examples

1. a. Determine the absolute extrema for \( f(x) = x^3 - 6x^2 + 1 \) on \([-2,3]\).

   Absolute maximum: \______________ \ Absolute minimum: \______________

   b. What if the interval is changed to \([-2,5]\)?

   Absolute maximum: \______________ \ Absolute minimum: \______________

2. Determine the absolute extrema for \( g(x) = x^{\frac{4}{3}} + 4x^{\frac{1}{3}} \) on \([-2,2]\).

   Absolute maximum: \______________ \ Absolute minimum: \______________
3. Determine the absolute extrema for \( p(x) = \frac{1}{2}(\sin^2 x + \cos x) + 2\sin x - x \) on \([0, 2\pi]\).

\[
p'(x) = \frac{1}{2}(2\sin x \cos x - \sin x) + 2\cos x - 1 = \frac{1}{2}(2\cos x - 1)(\sin x + 2)
\]

Absolute maximum: ____________________  Absolute minimum: ____________________

4. Determine the absolute extrema for \( h(x) = \frac{\ln x}{x} \) on \([1, 4]\).

Absolute maximum: ____________________  Absolute minimum: ____________________
IV. Relative Extrema / Local Extrema

A. Definitions
   a. A function \( f \) has a \textbf{relative maximum} value at an interior point \( c \) of its domain if \( f(x) \leq f(c) \) for all \( x \) in some open interval containing \( c \).
   b. A function \( f \) has a \textbf{relative minimum} value at an interior point \( c \) of its domain if \( f(x) \geq f(c) \) for all \( x \) in some open interval containing \( c \).

B. The First Derivative Test for Relative Extrema

Suppose that \( c \) is a critical point of a continuous function \( f \) and that \( f \) is differentiable at every point in some interval containing \( c \) except possibly at \( c \) itself.

1. If \( f'(x) > 0 \) on \((a,c)\) and \( f'(x) < 0 \) on \((c,b)\) then \( f \) has a relative maximum of \( f(c) \) at \( x=c \).
2. If \( f'(x) < 0 \) on \((a,c)\) and \( f'(x) > 0 \) on \((c,b)\) then \( f \) has a relative minimum of \( f(c) \) at \( x=c \).
3. If \( f' \) does not change signs at \( x=c \), then \( f \) has no relative extrema at \( x=c \).

C. Steps in using the First Derivative Test for Relative Extrema

1. Find \( f'(x) \)
2. Find the critical values. (Determine where \( f'(x) = 0 \) and/or \( f'(x) \) is undefined).
3. Determine the interval(s) where \( f \) is increasing (\( f'(x) > 0 \)) and interval(s) where \( f \) is decreasing (\( f'(x) < 0 \)).
4. a. If \( f \) is continuous, then \( f(c) \) is a relative maximum if \( f \) is increasing on \((a,c)\) followed by \( f \) decreasing on \((c,b)\).
   b. If \( f \) is continuous, then \( f(c) \) is a relative minimum if \( f \) is decreasing on \((a,c)\) followed by \( f \) increasing on \((c,b)\).

D. Examples

1. Determine the \( x \)-values where the function below on the interval, \([-5,10]\) has relative extrema.

![Graph of a function with relative extrema at x-values]

Relative maximum @\( x= \)

Relative minimum @ \( x= \)
2. For the following functions, determine (a) the intervals where the function is increasing and decreasing and (b) the relative extrema.

a. \( f(x) = x^2 \)
   - \( f \) increases on ________________
   - \( f \) decreases on ________________
   - Relative maximum: ________________
   - Relative minimum: ________________

b. \( y = (2x - 1)^3 \)
   - \( f \) increases on ________________
   - \( f \) decreases on ________________
   - Relative maximum: ________________
   - Relative minimum: ________________

c. \( g(t) = \frac{1}{4}t^4 + \frac{1}{2}t^3 - 5t^2 \)
   - \( f \) increases on ________________
   - \( f \) decreases on ________________
   - Relative maximum: ________________
   - Relative minimum: ________________
d. \( h(x) = x - 2\sqrt{x} \)  Note: domain is \([0, \infty)\)

\( f \) increases on ____________

\( f \) decreases on ____________

Relative maximum: ____________

Relative minimum: ____________

e. \( f(x) = xe^x \)

\( f \) increases on ____________

\( f \) decreases on ____________

Relative maximum: ____________

Relative minimum: ____________

f. Is there a relative minimum or maximum at \( x = 3.5 \)?