I. Review
   A. The average rate of change of \( y = f(x) \) wrt \( x \) over the interval \([x, x+h]\) is
      \[
      \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x+h) - f(x)}{h} , \text{ where } h = \Delta x.
      \]
   B. Graphically, this is the slope of the secant line between the points \((x, f(x))\) and \((x+h, f(x+h))\)

II. Tangent Line
   A. What is a Tangent Line?
      1. Wrt a Circle:  (1) A line is tangent to a circle at a pt \( P \) if the line \( L \) passes through \( P \) \( \perp \) to the radius at pt \( P \).  (2) Euclid states that a tangent is a line that intersects the circle once and only once.
      2. Wrt a Line: The tangent is the line itself
      3. Wrt a Curve:  (1) “Draw the best circle inside of the curve” (circle of curvature), (2) a tangent line can touch the curve in more than one place, (3) Def\( ^2 \): The tangent line to the curve \( y = f(x) \) at the pt \( P \) is the line \( L \) though \( P \) whose slope is the limit as pt \( Q \) approaches pt \( P \) of the slope of the secant line (from either side)
B. Calculating the Tangent Line

Using the Function: Pick a second pt \( Q \) on the graph of the fn \( y = f(x) \) close to pt \( P \).

Connect the points to form the secant line. As the change in \( x \) gets very small (close to zero) the secant line will become the tangent line. The main drawback to this method is if the curve takes a drastic turn.

** Make a chart to determine the slope of the secant lines as pt \( Q \) gets closer to pt \( P \).

Pt \( Q \) approaching \( P \) from the right. (Secant line in red, tangent line in blue)

Pt \( Q \) approaching \( P \) from the left.

Pt \( Q \) approaching \( P \) from the right.
C. Slope of Tangent Line

1. The **slope of the tangent line** to \( y = f(x) \) at the point \( P \ ((a, f(a)) \) is the number \( m_{\tan} \) given by:

\[
m = \lim_{x \to a} \left( \frac{f(x) - f(a)}{x - a} \right), \quad \text{provided this limit exists.} \tag{1}
\]

**OR**

\[
m = \lim_{h \to 0} \left( \frac{f(a + h) - f(a)}{h} \right), \quad \text{provided this limit exists.} \tag{2}
\]

2. A vertical tangent exists at \( x=a \) if the above limit goes to \( \pm \infty \) and the function is defined at \( x=a \).

3. A horizontal tangent exists at \( x=a \) if the above limit is 0.

4. The tangent line to the curve at the point \( P \ ((a, f(a)) \) is the line through \( P \) with the slope \( m \) defined above.

III. Instantaneous Rate of Change

A. Definitions

1. The instantaneous rate of change of \( y = f(x) \) is defined as

\[
f'(x) = \lim_{h \to 0} \left( \frac{f(x + h) - f(x)}{h} \right) = \lim_{\Delta x \to 0} \left( \frac{\Delta y}{\Delta x} \right), \quad \text{provided this limit exists.}
\]

2. Derivative \( = f'(x) = \lim_{h \to 0} \left( \frac{f(x + h) - f(x)}{h} \right) \)

3. The slope of the curve \( y = f(x) \) at the point \( P \ ((a, f(a)) \) is the number \( m \) given by:

\[
m = \lim_{h \to 0} \left( \frac{f(a + h) - f(a)}{h} \right), \quad \text{provided this limit exists.}
\]

4. Slope of the tangent line at the point \( P \ ((a, f(a)) \) is:

\[
m = \lim_{h \to 0} \left( \frac{f(a + h) - f(a)}{h} \right), \quad \text{provided this limit exists.}
\]
B. Examples

1. Calculating the Instantaneous Rate of Change (derivative) using the Average Rate of Change

Example a:

<table>
<thead>
<tr>
<th>h</th>
<th>1</th>
<th>0.1</th>
<th>0.01</th>
<th>0.001</th>
<th>0.0001</th>
</tr>
</thead>
<tbody>
<tr>
<td>Av. Rate of change of g over [7,7+h]</td>
<td>4</td>
<td>4.8</td>
<td>4.98</td>
<td>4.998</td>
<td>4.9998</td>
</tr>
<tr>
<td>h</td>
<td>-1</td>
<td>-0.1</td>
<td>-0.01</td>
<td>-0.001</td>
<td>-0.0001</td>
</tr>
<tr>
<td>Av. Rate of change of g over [7+h,7]</td>
<td>5</td>
<td>5.5</td>
<td>5.03</td>
<td>5.003</td>
<td>5.0003</td>
</tr>
</tbody>
</table>

\[ g'(7) = \ldots \]

Example b:

<table>
<thead>
<tr>
<th>h</th>
<th>1</th>
<th>0.1</th>
<th>0.01</th>
<th>0.001</th>
<th>0.0001</th>
</tr>
</thead>
<tbody>
<tr>
<td>Av. Rate of change of s over [0,h]</td>
<td>-2.52</td>
<td>-1.13</td>
<td>-0.6014</td>
<td>-0.6000144</td>
<td>-0.600001444</td>
</tr>
<tr>
<td>h</td>
<td>-1</td>
<td>-0.1</td>
<td>-0.01</td>
<td>-0.001</td>
<td>-0.0001</td>
</tr>
<tr>
<td>Av. Rate of change of s over [h,0]</td>
<td>-0.4</td>
<td>-0.598</td>
<td>-0.5998</td>
<td>-0.599982</td>
<td>-0.59999822</td>
</tr>
</tbody>
</table>

\[ s'(0) = \ldots \]
2. Determine whether the graph of the line to the curve below is an example of average rate of change or instantaneous rate of change.

3. Between which pairs of consecutive points in the graph below is the average rate of change:
   (Hint: think about secant lines)
   a. positive: ________________________
   b. negative: ________________________
   c. zero: ________________________

4. Between which pairs of consecutive points in the graph above is the instantaneous rate of change:
   (Hint: think about tangent lines)
   a. positive: ________________________
   b. negative: ________________________
   c. zero: ________________________
5. Given \( f(x) = \frac{1}{x^2}, x = 1 \). Find \( f'(1) \) using a table of average value rates.

<table>
<thead>
<tr>
<th>( h )</th>
<th>([1, 1+h] )</th>
<th>( \frac{f(x+h)-f(x)}{h} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>([1,2])</td>
<td>( \frac{f(2)-f(1)}{1} )</td>
</tr>
<tr>
<td>0.1</td>
<td>([1,1.1])</td>
<td>( \frac{f(1.1)-f(1)}{0.1} )</td>
</tr>
<tr>
<td>0.01</td>
<td>([1,1.01])</td>
<td>( \frac{f(1.01)-f(1)}{0.01} )</td>
</tr>
<tr>
<td>0.001</td>
<td>([1,1.001])</td>
<td>( \frac{f(1.001)-f(1)}{0.001} )</td>
</tr>
</tbody>
</table>

As \( h \to 0^+ \Rightarrow f'(1) = \) ___________

<table>
<thead>
<tr>
<th>( h )</th>
<th>([1+h,1])</th>
<th>( \frac{f(x+h)-f(x)}{h} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>([0,1])</td>
<td>( \frac{f(1)-f(0)}{1} )</td>
</tr>
<tr>
<td>-0.1</td>
<td>([0.9,1])</td>
<td>( \frac{f(1.1)-f(1)}{0.1} )</td>
</tr>
<tr>
<td>-0.01</td>
<td>([0.99,1])</td>
<td>( \frac{f(1.01)-f(1)}{0.01} )</td>
</tr>
<tr>
<td>-0.001</td>
<td>([0.999,1])</td>
<td>( \frac{f(1.001)-f(1)}{0.001} )</td>
</tr>
</tbody>
</table>

As \( h \to 0^- \Rightarrow f'(1) = \) ___________

Thus \( f'(1) = \) ___________

6. What is the equation of the tangent line to the graph of \( f(x) = \frac{1}{x^2} \) at \( x = 1 \)?

\[ m_{\text{tangent}} = \quad \quad \quad \quad \quad \quad \quad \quad f(1) = \quad \quad \quad \quad \quad \quad \quad \quad \]

IV. Summary

A. The following statements all refer to the same thing.

1. The derivative of \( f(x) \) at \( x = a \), \( f'(a) \)

2. The rate of change of \( f(x) \) with respect to \( x \) at \( x = a \)

3. The slope of the tangent to \( y = f(x) \) at \( x = a \)

4. The slope of \( y = f(x) \) at \( x = a \)

5. The limit of the difference quotient, \( \lim_{h \to 0} \left( \frac{f(a+h)-f(a)}{h} \right) \)

6. Velocity of \( y = f(t) \) at \( t = a \)

B. The tangent line to \( y = f(x) \) at \( (a,f(a)) \) is the line through \( (a,f(a)) \) whose slope is equal to \( f'(a) \), the derivative of \( f \) at \( a \).