Math 1014: Precalculus with Transcendental
Ch. 2: Functions and Graphs
Sec. 2.7 Inverse Functions

I. Inverse Functions
   A. Definition of the Inverse of a Function
      Let $f$ and $g$ be two functions such that
      
      $f(g(x)) = x$ for every $x$ in the domain of $g$
      
      and
      
      $g(f(x)) = x$ for every $x$ in the domain of $f$.

      The function $g$ is the inverse of the function $f$ and is denoted by $f^{-1}$ (read “$f$-inverse”).
      Thus, $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$. The domain of $f$ is equal to the range of $f^{-1}$, and vice versa.

      **NOTE:** $f^{-1} \neq \frac{1}{f}$

   B. Verifying Inverse Functions
      1. To verify if functions are inverses of each other, show that $f(f^{-1}(x)) = x$ and
         $f^{-1}(f(x)) = x$.

      2. Examples:
         Determine whether $f$ and $g$ are inverses of the other.

         a. $f(x) = 3x - 7$ and $g(x) = \frac{x + 3}{7}$

         b. $f(x) = \sqrt{x - 4}$ and $g(x) = x^3 + 4$
C. Horizontal Line Test

1. Horizontal Line Test
   A function \( f \) has an inverse that is a function \( f^{-1} \), if there is no horizontal line that intersects the graph of the function \( f \) at more than one point.

2. Examples
   Determine if each of the graphs represents a function that has an inverse function.

   a. \( f(x) = x + 2 \)
   b. \( f(x) = x^2 - 1 \)
   c. \( x^2 + y^2 = 4 \)

D. One-to-One Functions

1. Definition: A function is said to be one to one (1 - 1) if no two ordered pairs have the same second component but different first component, i.e., if \( x_1 \neq x_2 \), then \( f(x_1) \neq f(x_2) \).

   NOTE: A function has one \( y \) value for each \( x \) value but those \( y \) values can repeat.
   In a 1 – 1 function the \( y \) values never repeat.

2. Examples
   \( f(x) = x^2 \) is not 1 - 1
   \( f(x) = x^3 \) is 1 - 1

3. An equation must pass the Vertical Line Test to be a Function.
   A function must pass the Horizontal Line Test to be 1 - 1.

4. If \( f \) is not 1-1, then \( f^{-1} \) does not exist.
E. Finding the Inverse of a Function.

1. Graphically
   a. Steps
   
   Since the domain of \( f \) is the range of \( f^{-1} \) and the range of \( f \) is the domain of \( f^{-1} \). Interchange \( x \) and \( y \) coordinates of \( f \) in order to graph \( f^{-1} \).

   * The graph of \( f^{-1} \) can be found by reflecting the graph of \( f \) over the line \( y = x \).

   b. Example
   
   1) The points \((0,0), (1,1), (2,4), (3,9), \) etc. are on the graph of \( f(x) = x^2, x \geq 0 \).
   
   Interchanging the \( x \) and the \( y \) values gives the coordinates of \( f^{-1}(x) = \sqrt{x} \), \( (0,0), (1,1), (4,2), (9,3), \) etc.

   ![Graph of f and f^{-1}](image)

   2) If \( f \) has an inverse and the 4 points, \((0,1), (2,7), (5,20), (7,100), \) are on \( f \), what points must be on the inverse?

   ![Graph of f and f^{-1}](image)

   3) The graph of \( f' \) is below. Graph the inverse of \( f' \) on the same set of axes.

   ![Graph of f and f^{-1}](image)
2. Algebraically
   a. Steps
      The equation for the inverse of a function \( f \) can be found as follows:
      1) Replace \( f(x) \) with \( y \) in the equation for \( f(x) \).
      2) Interchange \( x \) and \( y \).
      3) Solve for \( y \). If this equation does not define \( y \) as a function of \( x \), the function \( f \) does not have an inverse function and this procedure ends. If this equation does define \( y \) as a function of \( x \), the function \( f \) has an inverse function.
      4) If \( f \) has an inverse function, replace \( y \) in step 3 by \( f^{-1}(x) \). We can verify our result by showing that \( f(f^{-1}(x)) = x \) and \( f^{-1}(f(x)) = x \).

b. Example
   Given \( f(x) = 2 + \sqrt{4 - x} \).
   1) Graph \( f \).

   ![Graph of f(x)](image)

2) State the domain and range of \( f \).
   Domain of \( f \): ________________  Range of \( f \): ________________
3) Find the inverse of \( f(x) = 2 + \sqrt{4-x} \)

4) State the domain and range of \( f^{-1} \).

   Domain of \( f^{-1} \): ________________  Range of \( f^{-1} \): ________________

5) Verify that \( f(f^{-1}(x)) = x \).
6) Verify that \( f^{-1}(f(x)) = x \).

7) Graph \( f^{-1} \) on the same set of axes as \( f \).