First Order Non-Linear Equations

We will briefly consider non-linear equations. In general, these may be much more difficult to solve than linear equations, but in some cases we will still be able to solve the equations. We will also show that solutions for an autonomous equation can be translated parallel to the t-axis.

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1 Theory of First Order Non-linear Equations

We have already seen that under somewhat reasonable conditions (such as continuous functions for coefficients) that linear first order initial value problems have unique solutions. In fact, we have shown how to find the solution using an integrating factor. This fixed approach to solving the equations is in fact how we can show that there is a unique solution. No such fixed process exists for non-linear equations. However, there are conditions under which it has been shown that a unique solution exists:

**Theorem 2.2:** (p. 49) Let \( R \) be the open rectangle defined by \( a < t < b, \alpha < y < \beta \). Let \( f(t, y) \) be a function of two variables defined on \( R \), where \( f(t, y) \) and the partial derivative \( \partial f / \partial y \) are continuous on \( R \). Suppose \((t_0, y_0)\) is a point in \( R \). Then there is an open \( t \)-interval \((c, d)\), contained in \((a, b)\) and containing \( t_0 \), in which there exists a unique solution to the initial value problem

\[
y' = f(t, y), \quad y(t_0) = y_0.
\]

**Example:**
We have already seen that \( y' = \sqrt{y}, \ y(0) = 0 \) does not have a unique solution. (Both \( y = 0 \) and \( y = t^2/4 \) are solutions.) Theorem 2.2 does not apply to this IVP because
\( \partial(\sqrt{y})/\partial y \) is discontinuous at \( y = 0 \).

Theorem 2.1 for linear first order equations was helpful in identifying the interval of existence. Unfortunately, the second theorem is less useful for determining the interval of existence. We can find a rectangle in which the solution will exist, but if we cannot solve the equation, it is difficult to know what \( t \) values will be valid.

**Example:**

Find the largest open rectangle in the plane in which the hypotheses of Theorem 2.2 are satisfied for:

\[
3ty' + 2\cos(y) = 1, \quad y(\pi/2) = -1
\]

The correct form is

\[
y' = \frac{1}{3t} - \frac{2\cos(y)}{3t}
\]

Now we need to compute \( \partial f / \partial y \):

\[
\frac{\partial}{\partial y} \left( \frac{1}{3t} - \frac{2\cos(y)}{3t} \right) = \frac{2\sin(y)}{3t}.
\]

Both \( f(t, y) \) and \( f_y(t, y) \) are discontinuous at \( t = 0 \). Neither are discontinuous anywhere else, so the largest rectangle is \( 0 < t < \infty, -\infty < y < \infty \).

**Example:**

Find the largest open rectangle in the plane in which the hypotheses of Theorem 2.2 are satisfied for:

\[
y' = \frac{-2t}{1 + y^3}, \quad y(1) = 1
\]

\( f(t, y) \) is:

\[
f(t, y) = \frac{-2t}{1 + y^3}
\]

\( \partial f / \partial y \) is:

\[
\frac{\partial f}{\partial y} = \frac{6ty^2}{(1 + y^3)^2}
\]

Both \( f(t, y) \) and \( f_y(t, y) \) are discontinuous at \( y = -1 \). Both are continuous for \( -\infty < t < \infty \). So the largest rectangle is \( -\infty < t < \infty, -1 < y < \infty \).
2 Autonomous Equations

Example:
Find the largest t-interval in which there exists a solution:

\[ y' = y^2, \quad y(0) = 1 \]

\[ f(y) = y^2 \]
\[ \frac{\partial f}{\partial y} = 2y \]

So the conditions of Theorem 2.2 are satisfied on the open rectangle \( R \) defined by \(-\infty < t < \infty, -\infty < y < \infty\). However, Theorem 2.2 doesn’t give the interval of existence of the solution. The unique solution is:

\[ y = \frac{1}{1-t} \]

Now we are able to restrict down the t-interval of existence which is \(-\infty < t < 1\).

There is a particular point regarding autonomous equations. Because autonomous equations depend only on \( y \) and not \( t \), it turns out that solutions to autonomous equations can be “translated” left or right. So the interval of existence is only dependent on how far we go from the initial starting point, not on where we started.

**Theorem 2.3:** (p. 51) Let the I.V.P

\[ y' = f(y), \quad y(0) = y_0. \]

satisfy the conditions of Theorem 2.2, and let \( y_1(t) \) be the unique solution, where the interval of existance for \( y_1(t) \) is \( a < t < b \), with \( a < 0 < b \).

Consider the I.V.P

\[ y' = f(y), \quad y(t_0) = y_0. \]

Then the function \( y_2(t) \) defined by \( y_2(t) = y_1(t - t_0) \) is the unique solution of the I.V.P and has an interval of existence \( t_0 + a < t < t_0 + b \).

In other words: Suppose we have the autonomous equation \( y' = f(y) \). Suppose also that \( y_1(t) \) is a solution to \( y' = f(y) \) with initial condition \( y(t_0) = y_0 \), and that the solution exists on the interval \( a < t < b \). If we then wish to solve the initial value problem \( y' = f(y) \) with \( y(t_0 + h) = y_0 \), we can just translate \( y_1(t) \): \( y_1(t - h) \) is a solution to the initial value problem. Note that the derivative will not change, so it
is still a solution to the differential equation and satisfy the initial condition; if we plug in \( t_0 + h \) we get:

\[
y_1([t_0 + h] - h) = y_1(t_0) = y_0
\]
as we wished. Of course, if \((t)\) exists on \((a, b)\), then \(y_1(t - h)\) exists on \((a + h, b + h)\).

A picture of the direction field for \(y' = y(2 - y)\) is shown below, together with the solutions to the initial value problems \(y' = y(2 - y)\), for \(y(2) = 0.25\), \(y(4) = 0.25\), and \(y(6) = 0.25\). Note that the solutions are just horizontal translations of each other!

To solve this equation:

\[
y' - 2y = -y^2
\]
look for a change of dependent variable: \(v(t) = (y(t))^{-1}\), using the chain rule then we have:

\[
dv/dt = -y^{-2}dy/dt
\]
and therefore:

\[
dy/dt = -y^2dv/dt = -v^{-2}dv/dt
\]
The differential equation then transforms into:

\[
dv/dt + 2v = 1
\]
and then reduces to the first order linear equation:

\[
v' + 2v = 1
\]
We can then solve this equation (I.F. = \(e^{\int 2 dt} = e^{2t}\)):

\[
v = 0.5 + Ce^{-2t}
\]
By substituting \(v(t) = (y(t))^{-1}\) into the equation, we get:

\[
y = 2/(1 + 2Ce^{-2t})
\]
3 Separable Equations

A separable differential equation is one that can be rewritten into the form

\[ A(y) \, dy = B(x) \, dx \]

where \( A \) is a function of \( y \) only and \( B \) a function of \( x \) only, and \( dx \) and \( dy \) are differentials. Thus we could also write \( A(y)y' = B(x) \). An alternate form (seen in the textbook) is the following:

\[ B(x) + A(y) \frac{dy}{dx} = 0 \]

(Here, there is obviously a difference of sign in the functions \( A(y) \) in the two forms.)

When an equation is separated like this, it is then possible to integrate directly on both sides of the equality to find a solution, which may be given as an implicit function.

4 Worked Examples: Separable Equations

Some non-linear equations are separable, which means this new technique will allow us to solve some problems we could not before.

example:

The differential equation \( 9y^2y' = 2x \) is nonlinear, but it is separable; we can rewrite it as

\[ 9y^2 \frac{dy}{dx} = 2x \]
\[ 9y^2 \, dy = 2x \, dx \]

Then integrating yields

\[ \int 9y^2 \, dy = \int 2x \, dx \]
\[ 3y^3 = x^2 + C \]

(Although both integrals would yield a constant of integration, we can combine the two constants into one.)

This gives a family of solutions dependent on the parameter \( C \).

Now the function \( y \) is given implicitly. We could solve for \( y \) to make the solution explicit:

\[ y = \sqrt[3]{\frac{1}{3}(x^2 + C)} \]
We can of course, solve for the constant \( C \) if we wish. (We can do this with the equation in either the implicit or explicit form.)

**Example:**

Solve the initial value problem

\[
y' = \frac{2x + 1}{y + 2}, \quad \text{and} \quad y(0) = 2
\]

We separate:

\[
(y + 2) \, dy = (2x + 1) \, dx
\]

and integrate:

\[
\frac{1}{2} y^2 + 2y = x^2 + x + C
\]

Since \( y(0) = 2 \), we plug in \( x = 0 \) and \( y = 2 \) to find \( C \):

\[
\frac{1}{2} \cdot 2^2 + (2)(2) = 6 \quad \text{and} \quad 0^2 + 0 + C = C, \quad \text{so} \quad C = 6.
\]

Finally, we solve for \( y \), which is not too difficult here. We proceed by multiplying through by 2 and completing the square:

\[
\frac{1}{2} y^2 + 2y = x^2 + x + 6
\]

\[
y^2 + 4y = 2x^2 + 2x + 12 \quad \text{(multiply by 2)}
\]

\[
y^2 + 4y + 4 = 2x^2 + 2x + 16 \quad \text{(completing the square)}
\]

\[
(y + 2)^2 = 2x^2 + 2x + 16 \quad \text{(factoring)}
\]

So finally we get

\[
y = \sqrt{2x^2 + 2x + 16} - 2.
\]

(We know that it is the positive square root, because we must have \( y(0) = 2 \), which is positive.)

We make a further note about our solution: since \( 2x^2 + 2x + 16 > 0 \) for all values of \( x \), the solution is valid for all \( x \). If it were not, we would need to restrict the values of \( x \) in the solution. Further, note that our function never has \( y = -2 \), so the function is always a solution to the original differential equation which has \( y + 2 \) in the denominator of the right hand side.

Some first order equations may be both separable and linear, and so may be solved different ways:

**Example:**
\( y' = x - 1 + xy - y \) is a linear differential equation, as can be seen by rewriting it as \( y' = (x - 1) + (x - 1)y \). We could use an integrating factor \( \exp(x - x^2/2) \) for the equation in the form \( y' + (1 - x)y = x - 1 \), but the equation is also separable:

\[ y' = (x - 1) + (x - 1)y = (x - 1)(y + 1) \]

So we get

\[ \frac{y'}{y + 1} = x - 1, \quad \text{or} \quad \frac{dy}{y + 1} = (x - 1)dx \]

Then we can integrate on both sides to get \( \ln |y + 1| = \frac{1}{2}x^2 - x + C \), or (by solving) the equivalent \( y = A \exp(\frac{1}{2}x^2 - x) - 1 \), where \( A = \pm e^C \).

Not all first order linear equations are separable:

**Example:**

\( y' = 3y + e^x \) is not separable. Note that if we rewrite it using differentials, the best we can do is

\[ dy = (3y + e^x) \, dx \]

which is not separated. However, if we rewrite the equation as \( y' - 3y = e^x \), we could solve the equation by an integrating factor of \( e^{-3x} \).

Some equations are neither linear nor separable:

**Example:**

The equation below is neither linear nor separable:

\[ y' = \frac{x + y}{x - y} \]

If we attempt to separate it, we get only \((x - y)dy = (x + y)dx\).

As we saw above, sometimes a separable equation gives a result that is an implicit function. If it’s not too hard to solve for \( y \), go ahead. Otherwise, it might be best to leave it in implicit form.

**Example:**

Find a solution to the initial value problem \( y' = \cos^2(x) \cos^2(2y) \), \( y(\pi/2) = \pi \):

We start by separating and integrating:

\[ \int \frac{dy}{\cos^2(2y)} = \int \cos^2(x) \, dx \]
The left hand side is \( \int \sec^2(2y) \, dy = \frac{1}{2} \tan(2y) + C \), while the right side gives

\[
\int \cos^2(x) \, dx = \int \frac{1}{2} [1 + \cos(2x)] \, dx = \frac{1}{2} x + \frac{1}{4} \sin(2x) + C.
\]

Thus, we have

\[
\tan(2y) = x + \frac{1}{2} \sin(2x) + C
\]

Now we need to know find a value of \( C \) which can satisfy the initial condition \( y(\pi/2) = \pi \). That is, \( y = \pi \) when \( x = \pi/2 \), so we see that we have \( \tan(2\pi) = \pi/2 + \frac{1}{2} \sin(2\pi/2) + C \), or \( 0 = \pi/2 + C \), so \( C = -\pi/2 \).

So the solution to the initial value problem is

\[
\tan(2y) = x + \frac{1}{2} \sin(2x) - \pi/2.
\]

Is it reasonable to solve for \( y \)? If we use the arctangent, we have the problem that the range of the standard branch of the arctangent has range \((-\pi/2, \pi/2)\), so that we cannot have \( y(\pi/2) = \pi \)! Therefore, we might be better off leaving this solution as implicit.

Note that we could write down a solution by adjusting the arctangent. In fact, given that \( \tan(x) \) repeats every \( \pi \), an explicit solution is given by the following:

\[
y = \frac{1}{2} \left[ \tan^{-1} \left( x + \frac{1}{2} \sin(2x) - \pi/2 \right) + 2\pi \right]
\]