Solutions 3.4

2. a) The characteristic equation is

$$9\lambda^2 - 6\lambda + 1 = 0,$$

leading to $\lambda = 1/3$. The general solution is

$$y = c_1 e^{t/3} + c_2 t e^{t/3}.$$

b)

$$y(3) = ec_1 + 3ec_2 = -2, \quad y'(3) = \frac{e}{3}c_1 + 2ec_2 = -5/3,$$

hence $c_1 = 1/e$, $c_2 = -1/e$.
c) $\lim_{t\to-\infty} y(t) = 0$, $\lim_{t\to\infty} y(t) = -\infty$.
6. a) The characteristic equation is

$$\lambda^2 - 4\lambda + 4 = 0,$$

leading to $\lambda = 2$. The general solution is

$$y = c_1 e^{2t} + c_2 t e^{2t}.$$

b)

$$y(-1) = e^{-2}c_1 - e^{-2}c_2 = 2, \quad y'(-1) = 2e^{-2}c_1 - e^{-2}c_2 = 1,$$

hence $c_1 = -e^2$, $c_2 = -3e^2$.
c) $\lim_{t\to-\infty} y(t) = 0$, $\lim_{t\to\infty} y(t) = -\infty$.

Solutions 3.5

1. a) $2(\cos(\pi/3) + i\sin(\pi/3)) = 1 + i\sqrt{3}$.
b) $-2\sqrt{2}(\cos(\pi/4) - i\sin(\pi/4)) = -2 + 2i$.
c) $(2 - i)(-i) = -1 - 2i$.
d) $1/(2\sqrt{2})(-\sqrt{3}/2 - i/2) = -\sqrt{3}/(4\sqrt{2}) - i/(4\sqrt{2})$.
e) $4\exp(2\pi i/3) = 4(-1/2 + i\sqrt{3}/2) = -2 + 2i\sqrt{3}$.
5. a) $9\lambda^2 + 1 = 0$, i.e. $\lambda = \pm i/3$.
b) $y = c_1\cos(t/3) + c_2\sin(t/3)$.
c)

$$y(\pi/2) = \frac{\sqrt{3}}{2}c_1 + \frac{1}{2}c_2 = 4,$$

$$y'(\pi/2) = -\frac{1}{6}c_1 + \frac{\sqrt{3}}{6}c_2 = 0.$$

This leads to $c_2 = 2$, $c_1 = 2\sqrt{3}$.
12. a) $9\lambda^2 + \pi^2 = 0$, i.e. $\lambda = \pm i\pi/3$.
b) $y = c_1 \cos(\pi t/3) + c_2 \sin(\pi t/3)$.
c)

$$y(3) = -c_1 = 2,$$
$$y'(3) = -\frac{\pi}{3}c_2 = -\pi.$$

This leads to $c_1 = -2$, $c_2 = 3$.

Solutions 3.7

1. a)

$$y_p' = 3, \quad y_p'' = 0, \quad y_p'' - 2y_p' - 3y_p = -6 - 3(3t-1) = -9t - 3.$$

b) The characteristic equation is

$$r^2 - 2r - 3 = 0,$$

which has roots -1 and 3. Hence

$$y_c = c_1 e^{-t} + c_2 e^{3t}.$$

c) The general solution is

$$y = 3t - 1 + c_1 e^{-t} + c_2 e^{3t}.$$

Hence

$$y(0) = -1 + c_1 + c_2 = 1,$$
$$y'(0) = 3 - c_1 + 3c_2 = 3.$$

This leads to $c_1 = 3/2$, $c_2 = 1/2$.
6. a)

$$y_p' = -2e^{-t} + 2te^{-t}, \quad y_p'' = 4e^{-t} - 2te^{-t}, \quad y_p'' + y_p' = 2e^{-t}.$$

b) The characteristic equation is

$$r^2 + r = 0,$$

which has roots -1 and 0. Hence

$$y_c = c_1 e^{-t} + c_2.$$

c) The general solution is

$$y = -2te^{-t} + c_1 e^{-t} + c_2.$$

Hence

$$y(0) = c_1 + c_2 = 2,$$
$$y'(0) = -2 - c_1 = 2.$$

This leads to $c_1 = -4$, $c_2 = 6$.

10. a)

$$y_p' = -2\sin t + \cos t, \quad y_p'' = -2\cos t - \sin t, \quad y_p'' - 2y_p' + 2y_p = 5\sin t.$$

b) The characteristic equation is

$$r^2 - 2r + 2 = 0,$$

which has roots $1 \pm i$. Hence

$$y_c = c_1 e^t \cos t + c_2 e^t \sin t.$$

c) The general solution is

$$y = 2\cos t + \sin t + c_1 e^t \cos t + c_2 e^t \sin t.$$

Hence

$$y(\pi/2) = 1 + c_2 e^{\pi/2} = 1,$$
$$y'(\pi/2) = -2 - c_1 e^{\pi/2} + c_2 e^{\pi/2} = 0.$$

This leads to $c_1 = -2e^{-\pi/2}$, $c_2 = 0$.

Solutions 3.8

1. a) The characteristic equation is $r^2 - 4 = 0$ with roots ± 2. So the complementary solution is

$$y_c = c_1 e^{2t} + c_2 e^{-2t}.$$

b) A particular solution has the form

$$y = At^2 + Bt + C,$$

leading to

$$y' = 2At + B, \quad y'' = 2A, \quad y'' - 4y = -4At^2 - 4Bt + (2A - 4C) = 4t^2.$$

Consequently, we must have

$$-4A = 4, \quad -4B = 0, \quad 2A - 4C = 0,$$

i.e. $A = -1$, $B = 0$, $C = -1/2$.
c)

$$y = -t^2 - \frac{1}{2} + c_1 e^{2t} + c_2 e^{-2t}.$$

3. a) The characteristic equation is $r^2 + 1 = 0$ with roots $\pm i$. So the complementary solution is

$$y_c = c_1 \cos t + c_2 \sin t.$$

b) A particular solution has the form

$$y = Ae^t,$$

leading to

$$y' = y'' = Ae^t, \quad y'' + y = 2Ae^t = 8e^t.$$

Consequently $A = 4$.
c)

$$y = 4e^t + c_1 \cos t + c_2 \sin t.$$

4. a) The characteristic equation is $r^2 + 1 = 0$ with roots $\pm i$. So the complementary solution is

$$y_c = c_1 \cos t + c_2 \sin t.$$

b) A particular solution has the form

$$y = Ae^t \cos t + Be^t \sin t,$$

leading to

$$y' = (A + B)e^t \cos t + (B - A)e^t \sin t, \quad y'' = 2Be^t \cos t - 2Ae^t \sin t,$$

$$y'' + y = (2B + A)e^t \cos t + (B - 2A)e^t \sin t = e^t \sin t.$$

Consequently, we must have

$$2B + A = 0, \quad B - 2A = 1,$$

i.e. $A = -2/5$, $B = 1/5$.
c)

$$y = -\frac{2}{5}e^t \cos t + \frac{1}{5}e^t \sin t + c_1 \cos t + c_2 \sin t.$$

5. a) The characteristic equation is $r^2 - 4r + 4 = 0$ with double root 2. So the complementary solution is

$$y_c = c_1 e^{2t} + c_2 t e^{2t}.$$

b) A particular solution has the form

$$y = At^2 e^{2t},$$

leading to

$$y' = (2At^2 + 2At)e^{2t}, \quad y'' = (4At^2 + 8At + 2A)e^{2t},$$

$$y'' - 4y' + 4y = 2Ae2t = e^{2t}.$$

Consequently, $A = 1/2$.
c)

$$y = \frac{1}{2}t^2 e^{2t} + c_1 e^{2t} + c_2 t e^{2t}.$$

14. a) The characteristic equation is $r^2 + 4r + 5 = 0$ with roots $-2 \pm i$. So the complementary solution is

$$y_c = c_1 e^{-2t} \cos t + c_2 e^{-2t} \sin t.$$

b) A particular solution has the form

$$y = At + B + Ce^{-t},$$

leading to

$$y' = A - Ce^{-t}, \quad y'' = Ce^{-t}, \quad y'' + 4y' + 5y = 5At + (4A + 5B) + 2Ce^{-t} = 5t + e^{-t}.$$

Consequently, we must have $A = 1$, $B = -4/5$, $C = 1/2$.
c)

$$y = t - \frac{4}{5} + \frac{1}{2}e^{-t} + c_1 e^{-2t}\cos t + c_2 e^{-2t}\sin t.$$

Solutions 3.9

1. a) The characteristic equation is $r^2 + 4 = 0$ with roots $\pm 2i$, so the complementary solution is

$$y_c = c_1 \cos(2t) + c_2 \sin(2t).$$

b) We look for a particular solution in the form

$$y_p = u_1(t)\cos(2t) + u_2(t)\sin(2t),$$

and obtain the equations

$$u_1' \cos(2t) + u_2' \sin(2t) = 0,$$

$$-2u_1' \sin(2t) + 2u_2' \cos(2t) = 4.$$

This leads to

$$u_1' = -2\sin(2t) =, \quad u_2' = 2\cos(2t).$$

Integration yields

$$u_1 = \cos(2t) + c_1, \quad u_2 = sin(2t) + c_2,$$

$$y_p = 1 + c_1 \cos(2t) + c_2 \sin(2t).$$

c) With undetermined coefficients, we have the particular solution

$$y_p = A,$$

leading to

$$y_p' = y_p'' = 0,$$
$$y_p'' + 4y_p = 4A = 4.$$

Hence $A = 1$.
2. a) The characteristic equation is $r^2 + 1 = 0$ with roots $\pm i$, so the complementary solution is

$$y_c = c_1 \cos t + c_2 \sin t.$$

b) We look for a particular solution in the form

$$y_p = u_1(t) \cos t + u_2(t) \sin t,$$

and obtain the equations

$$u_1' \cos t + u_2' \sin t = 0,$$
$$-u_1' \sin t + u_2' \cos t = \frac{1}{\cos t}.$$

This leads to

$$u_1' = -\tan t, \quad u_2' = 1.$$

Integration yields

$$u_1 = \ln \cos t, \quad u_2 = t.$$

Hence

$$y_p = \cos t \ln \cos t + t \sin t.$$

3. We look for a particular solution of the form

$$y_p = u_1(t)e^t + u_2(t)t^2 e^t.$$

This yields

$$u_1' e^t + u_2' t^2 e^t = 0,$$
$$u_1' e^t + u_2'(t^2 + 2t)e^t = te^t.$$

Consequently,

$$u_1' = -\frac{t^2}{2}, \quad u_2' = \frac{1}{2}.$$

Integration yields

$$u_1 = -\frac{t^3}{6}, \quad u_2 = \frac{t}{2},$$

and

$$y_p = \frac{t^3 e^t}{3}.$$

9. We look for a particular solution of the form

$$y_p = u_1(t)\sin t + u_2(t) t \sin t.$$

This yields

$$u_1' \sin t + u_2' t \sin t = 0,$$
$$u_1' \cos t + u_2'(\sin t + t\cos t) = t \sin t.$$

Consequently,

$$u_1' = -t^2, \quad u_2' = t.$$

Integration yields

$$u_1 = -\frac{t^3}{3}, \quad u_2 = \frac{t^2}{2},$$

and

$$y_p = \frac{t^3}{6}\sin t.$$

Solutions 3.10

1. a) The characteristic equation is $r^2 + \omega_0^2 = 0$ with roots $r = \pm i\omega_0$, so

$$y_c = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t).$$

b) (i) We have

$$y_p = A\cos(\omega t) + B\sin(\omega t),$$
$$y_p'' = -A\omega^2 \cos(\omega t) - B\omega^2 \sin(\omega t),$$
$$y_p'' + \omega_0^2 y_p = A(\omega_0^2 - \omega^2)\cos(\omega t) + B(\omega_0^2 - \omega^2)\sin(\omega t) = F\cos(\omega t),$$

hence $A = F/(\omega_0^2 - \omega^2)$, $B = 0$.
(ii) We have

$$y_p = At\cos(\omega_0 t) + Bt\sin(\omega_0 t),$$
$$y_p' = -\omega_0 At\sin(\omega_0 t) + \omega_0 Bt\cos(\omega_0 t) + A\cos(\omega_0 t) + B\sin(\omega_0 t),$$
$$y_p'' = -\omega_0^2 At\cos(\omega_0 t) - \omega_0^2 Bt\sin(\omega_0 t) - 2A\omega_0 \sin(\omega_0 t) + 2B\cos(\omega_0 t),$$
$$y_p'' + \omega_0^2 y = -2A\omega_0 \sin(\omega_0 t) + 2B\cos(\omega_0 t) = F\cos(\omega_0 t),$$

hence $A = 0$, $B = F/(2\omega_0)$.
4. The spring constant is

$$\frac{10 * 9.8}{0.098} = 1000 \frac{\text{kg}}{\text{sec}^2}.$$

Hence the differential equation is

$$10y'' + 1000y = 20\cos(8t), y(0) = 0,\ y'(0) = 0.$$

The solution is

$$y(t) = \frac{1}{18}(\cos(8t) - \cos(10t)),$$

(see formula (7)). We can rewrite the answer in the form

$$y(t) = \frac{1}{9}\sin(9t)\sin t,$$

this makes it evident that the maximum excursion from equilibrium is 1/9.
7. We have the differential equation

$$2y'' + 8y' + 80y = 20\cos(8t),\ y(0) = 0,\ y'(0) = 0.$$

We divide the differential equation by 2:

$$y'' + 4y' + 40y = 10\cos(8t).$$

The characteristic equation is $r^2 + 4r + 40 = 0$, which has roots $r = -2 \pm 6i$. Hence the complementary solution is

$$y_c = c_1 e^{-2t}\cos(6t) + c_2 e^{-2t}\sin(6t).$$

A particular solution can be found in the form

$$y_p = A\cos(8t) + B\sin(8t),$$

which leads to

$$y_p' = -8A\sin(8t) + 8B\cos(8t),$$

$$y_p'' = -64A\cos(8t) - 64B\sin(8t),$$

$$y_p'' + 4y_p' + 40y_p = (-24A + 32B)\cos(8t) + (-32A - 24B)\sin(8t) = 10\cos(8t).$$

Hence,

$$-24A + 32B = 10, \quad -32A - 24B = 0,$$

i.e. $A = -3/20$, $B = 1/5$. The general solution is now

$$y = -\frac{3}{20}\cos(8t) + \frac{1}{5}\sin(8t) + c_1 e^{-2t}\cos(6t) + c_2 e^{-2t}\sin(6t).$$

We find

$$y(0) = -\frac{3}{20} + c_1 = 0,$$

$$y'(0) = \frac{8}{5} - 2c_1 + 6c_2 = 0,$$

i.e. $c_1 = 3/20$, $c_2 = -13/60$. For $t \to \infty$, the complementary solution tends to zero, and the particular solution remains, which oscillates periodically.

Solutions 3.11

3. With

$$y = c_1 + c_2 t + c_3\cos(2t) + c_4\sin(2t),$$

we have

$$y' = c_2 - 2c_3\sin(2t) + 2c_4\cos(2t),$$

$$y'' = -4c_3\cos(2t) - 4c_4\sin(2t),$$

$$y''' = 8c_3\sin(2t) - 8c_4\cos(2t).$$

Hence,

$$y(0) = c_1 + c_3 = 0,$$

$$y'(0) = c_2 + 2c_4 = -1,$$

$$y''(0) = -4c_3 = -4,$$

$$y'''(0) = -8c_4 = 8.$$

The solution is $c_1 = -1$, $c_2 = 1$, $c_3 = 1$, $c_4 = -1$.
7. We have

$$W(1) = 2 * 2 - (-1) * (-4) = 0,$$

so the two solutions do not form a fundamental set.

18. The general solution is

$$y = c_1 e^{-2t} \cos t + c_2 e^{-2t} \sin t,$$

which leads to

$$y(0) = c_1, \quad y'(0) = -2c_1 + c_2.$$

For the given initial conditions, we find

$$\bar{y}_1(t) = e^{-2t} \cos t + e^{-2t} \sin t,$$

$$\bar{y}_2(t) = e^{-2t} \sin t.$$

Solutions 3.12

1. a) $\lambda^3 - 4\lambda = 0$.
b) $\lambda^3 - 4\lambda = \lambda(\lambda^2 - 4)$, so the roots are 0 and ± 2.
c) $y = c_1 + c_2 e^{2t} + c_3 e^{-2t}$.
2. a) $\lambda^3 + \lambda^2 - \lambda - 1 = 0$.
b) $\lambda^3 + \lambda^2 - \lambda - 1 = (\lambda + 1)(\lambda^2 - 1) = (\lambda + 1)^2(\lambda - 1)$, so the roots are -1 (double) and 1.
c) $y = c_1 e^t + c_2 e^{-t} + c_3 t e^{-t}$.
5. a) $16\lambda^4 + 8\lambda^2 + 1 = 0$.
b) $16\lambda^4 + 8\lambda^2 + 1 = (4\lambda^2 + 1)^2$, so the roots are $\pm i/2$, each double.
c) $y = c_1 \cos(t/2) + c_2 \sin(t/2) + c_3 t \cos(t/2) + c_4 t \sin(t/2)$.
8. a) $\lambda^4 - 1 = 0$.
b) $\lambda^4 - 1 = (\lambda^2 - 1)(\lambda^2 + 1)$, so the roots are ± 1, $\pm i$.
c) $y = c_1 e^t + c_2 e^{-t} + c_3 \cos(t) + c_4 \sin(t)$.

Solutions 4.1

6.

$$\det A = \begin{vmatrix} t+1 & t \\ t & t+1 \end{vmatrix} = (t+1)^2 - t^2 = 2t + 1.$$

Hence A is invertible unless $t = -1/2$. The inverse is

$$A^{-1} = \frac{1}{2t+1} \begin{pmatrix} t+1 & -t \\ -t & t+1 \end{pmatrix}.$$

8.

$$\det A = \begin{vmatrix} \sin t & -\cos t \\ \sin t & \cos t \end{vmatrix} = 2\sin t\cos t.$$

Hence A is invertible unless t is a multiple of $\pi/2$. The inverse is

$$A^{-1} = \frac{1}{2\sin t\cos t}\begin{pmatrix} \cos t & \cos t \\ -\sin t & \sin t \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \csc t & \csc t \\ -\sec t & \sec t \end{pmatrix}.$$

14.

$$P(t) = \begin{pmatrix} t^2 & 3 \\ \sin t & t \end{pmatrix}, \quad \mathbf{g}(t) = \begin{pmatrix} \sec t \\ -5 \end{pmatrix}.$$

15.

$$P(t) = \begin{pmatrix} t^{-1} & t^2+1 \\ 4 & t^{-1} \end{pmatrix}, \quad \mathbf{g}(t) = \begin{pmatrix} t \\ 8t\ln t \end{pmatrix}.$$

19. Integration yields

$$A(t) = \begin{pmatrix} \ln t & 2t^2 \\ 5t & t^3 \end{pmatrix} + C,$$

where C is a constant matrix. From the initial condition we find

$$C = A(1) - \begin{pmatrix} 0 & 2 \\ 5 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ -4 & -3 \end{pmatrix}.$$

Solutions 4.2

4. The singular points are at ± 2, so the interval of existence is $(-2,2)$.
9. We find

$$\mathbf{y}' = 2c_1e^{2t}\begin{pmatrix} 2 \\ -1 \end{pmatrix} + 3c_2e^{3t}\begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

$$A\mathbf{y} = c_1e^{2t}\begin{pmatrix} 4 \\ -2 \end{pmatrix} + c_2\begin{pmatrix} 3 \\ -3 \end{pmatrix}.$$

The two agree. To satisfy the initial condition, we want

$$c_1\begin{pmatrix} 2 \\ -1 \end{pmatrix} + c_2\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 4 \\ -3 \end{pmatrix},$$

i.e.

$$2c_1 + c_2 = 4, \quad -c_1 - c_2 = -3.$$

This leads to $c_1 = 1$, $c_2 = 2$.

12. We first divide by $\cos t$ to achieve standard form

$$y'' - \frac{3t}{\cos t}y' + \frac{\sqrt{t}}{\cos t}y = \frac{t^2+1}{\cos t}.$$

Then, as usual, we set $y_1 = y$, $y_2 = y'$. The system thus obtained is

$$y_1' = y_2,$$

$$y_2' = \frac{3t}{\cos t}y_2 - \frac{\sqrt{t}}{\cos t}y_1 + \frac{t^2+1}{\cos t}.$$

Consequently,

$$P(t) = \begin{pmatrix} 0 & 1 \\ -\frac{\sqrt{t}}{\cos t} & \frac{3t}{\cos t} \end{pmatrix}, \quad \mathbf{G}(t) = \begin{pmatrix} 0 \\ \frac{t^2+1}{\cos t} \end{pmatrix}.$$

13. We first divide by e^t to achieve standard form

$$y''' + 5e^{-t}y'' + t^{-1}e^{-t}y' + (\tan t)e^{-t}y = e^{-t}.$$

Then, as usual, we set $y_1 = y$, $y_2 = y'$, $y_3 = y''$. The system thus obtained is

$$y_1' = y_2,$$

$$y_2' = y_3,$$

$$y_3' = -5e^{-t}y_2 - t^{-1}e^{-t}y_2 - (\tan t)e^{-t}y_1 + e^{-t}.$$

Consequently,

$$P(t) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -(\tan t)e^{-t} & -t^{-1}e^{-t} & -5e^{-t} \end{pmatrix}, \quad \mathbf{G}(t) = \begin{pmatrix} 0 \\ 0 \\ e^{-t} \end{pmatrix}.$$

Solutions 4.4

1.

$$A = \begin{pmatrix} 5 & 3 \\ -6 & -4 \end{pmatrix}.$$

The characteristic polynomial is

$$(5-\lambda)(-4-\lambda) + 18 = \lambda^2 - \lambda - 2 = 0.$$

The eigenvalues are -1 and 2.

8.

$$A = \begin{pmatrix} 0 & 1 & -3 \\ 0 & -5 & -4 \\ 0 & 8 & 7 \end{pmatrix}.$$

The characteristic polynomial is

$$\begin{vmatrix} -\lambda & 1 & -3 \\ 0 & -5-\lambda & -4 \\ 0 & 8 & 7-\lambda \end{vmatrix}$$

$$= (-\lambda)((-5-\lambda)(7-\lambda)+32) = (-\lambda)(\lambda^2 - 2\lambda - 3) = 0.$$

The eigenvalues are 0, 3 and -1.

11. The equations for an eigenvector are

$$-6x_1 + 3x_2 = 0, \quad -4x_1 + 2x_2 = 0,$$

which simplifies to $x_2 = 2x_1$. Up to a constant multiple, the eigenvector is

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

17. We can use the following sequence of row reductions:

$$\begin{pmatrix} -2 & 3 & 1 \\ -8 & 13 & 5 \\ 11 & -17 & -6 \end{pmatrix},$$

$$\begin{pmatrix} 1 & -3/2 & -1/2 \\ -8 & 13 & 5 \\ 11 & -17 & -6 \end{pmatrix},$$

$$\begin{pmatrix} 1 & -3/2 & -1/2 \\ 0 & 1 & 1 \\ 0 & -1/2 & -1/2 \end{pmatrix},$$

$$\begin{pmatrix} 1 & -3/2 & -1/2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

The simplified system is $x_1+x_3=0$, $x_2+x_3=0$. If we arbritarily set $x_3=0$, we find the eigenvector

$$\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}.$$

21. The characteristic equation is $\lambda^2-1=0$, leading to eigenvalues ± 1. An eigenvector for 1 has to satisfy the equation $-x_1+x_2=0$, hence the eigenvector is

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

An eigenvector for -1 must satisfy $x_1+x_2=0$, leading to the eigenvector

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

We obtain the two linearly independent solutions

$$e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Solutions 4.5

3. The characteristic equation is

$$(-5-\lambda)(5-\lambda)+24=\lambda^2-1=0,$$

leading to eigenvalues ± 1. An eigenvector for 1 satisfies the equations

$$-6x_1-2x_2=0, \qquad 12x_1+4x_2=0,$$

which yield the eigenvector

$$\begin{pmatrix} 1 \\ -3 \end{pmatrix}.$$

The eigenvector for -1 satisfies

$$-4x_1-2x_2=0, \qquad 12x_1+6x_2=0,$$

which yield the eigenvector

$$\begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

Thus the general solution is

$$\mathbf{y} = c_1 e^t \begin{pmatrix} 1 \\ -3 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

We have

$$\mathbf{y}(1) = c_1 e \begin{pmatrix} 1 \\ -3 \end{pmatrix} + c_2 e^{-1} \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

i.e.

$$c_1 e + c_2 e^{-1} = 1,$$

$$-3c_1 e - 2c_2 e^{-1} = 0.$$

This yields $c_1 = -2/e$, $c_2 = 3e$.

8. The characteristic polynomial is given in the problem, it has roots 0 and 6. Eigenvalues for $\lambda = 0$ must satisfy the equation

$$2x_1 + 2x_2 + 2x_3 = 0.$$

This allows us to prescribe x_2 and x_3 arbitrarily, and then $x_1 = -x_2 - x_3$. The general eigenvector is

$$x_2 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$$

For $\lambda = 6$, we find the system

$$-4x_1 + 2x_2 + 2x_3 = 0, \qquad 2x_1 - 4x_2 + 2x_3 = 0, \qquad 2x_1 + 2x_2 - 4x_3 = 0,$$

which can be simplified to

$$x_1 = x_2 = x_3.$$

The corresponding eigenvector is therefore

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

The general solution of the system is therefore

$$\mathbf{y} = c_1 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + c_3 e^{6t} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

The initial condition leads to

$$-c_1 - c_2 + c_3 = 2, \qquad c_1 + c_3 = 5, \qquad c_2 + c_3 = 5.$$

The solution is $c_1 = c_2 = 1$, $c_3 = 4$.
13.a) We have $Q_1' = -2\frac{r}{V}Q_1 + \frac{r}{V}Q_2$, $Q_2' = \frac{r}{V}Q_1 - 2\frac{r}{V}Q_2$.
b) The matrix is

$$\begin{pmatrix} -2r/V & r/V \\ r/V & -2r/V \end{pmatrix},$$

which has characteristic equation $(-2r/V-\lambda)^2-(r/V)^2 = 0$, and eigenvalues $-3r/V$ and $-r/V$. The eigenvector for $-3r/V$ is

$$\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

and the eigenvector for r/V is

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

The general solution is therefore

$$\begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} = c_1 \exp(-3rt/V) \begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 \exp(-rt/V) \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

c) We have

$$Q(0) = \begin{pmatrix} Q_0 \\ 2Q_0 \end{pmatrix} = c_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

This yields $c_1 = Q_0/2$, $c_2 = 3Q_0/2$. Hence we find

$$Q_1(\tau) = -\frac{Q_0}{2}e^{-3\tau} + \frac{3Q_0}{2}e^{-\tau},$$

$$Q_2(\tau) = \frac{Q_0}{2}e^{-3\tau} + \frac{3Q_0}{2}e^{-\tau},$$

where $\tau = rt/V/$ Clearly, $Q_2 > Q_1$, and we want to find τ when

$$\frac{3Q_0}{2}e^{-\tau} + \frac{Q_0}{2}e^{-3\tau} = 0.01Q_0.$$

We can solve this approximately by assuming that $e^{-3\tau}$ is negligible. We then find

$$e^{-\tau} = \frac{2}{300},$$

which yields

$$\tau = \ln(150) = 5.010635.$$

(The exact solution is $\tau = 5.010651$). The corresponding t is

$$t = \frac{\tau V}{r} = 250.53\,\text{sec}.$$

17. For part a, the eigenvalues are -2 and 1 with corresponding eigenvectors $(1,1)$ and $(1,0)$. Hence the unstable direction must be along the x axis and the stable direction along the first diagonal. This is plot 4.
For part b, the eigenvalues are -2 and -1, so the picture is a stable node. This is plot 3.
For part c, the eigenvalues are 4 and 2, so the picture must be an unstable node. This is plot 2.
For part d, the eigenvalues are 2 and -1, with eigenvectors along the diagonals. This is plot 1.

Solutions 4.6

1. The characteristic equation is

$$(2-\lambda)^2 + 1 = 0,$$

which leads to $\lambda = 2 \pm i$. An eigenvector for $2+i$ must satisfy

$$2x_1 + x_2 = (2+i)x_1, \quad -x_1 + 2x_2 = (2+i)x_2.$$

This reduces to the single equation $x_2 = ix_1$. Therefore we have the eigenvector

$$\begin{pmatrix} 1 \\ i \end{pmatrix}.$$

The eigenvector for $2-i$ is the complex conjugate, i.e.

$$\begin{pmatrix} 1 \\ -i \end{pmatrix}.$$

9. The characteristic equation is

$$\begin{vmatrix} 2-\lambda & 2 & 9 \\ 1 & -1-\lambda & 3 \\ -1 & -1 & -4-\lambda \end{vmatrix}$$

$$= (2-\lambda)(-1-\lambda)(-4-\lambda) - 6 - 9 + 3(2-\lambda) + 9(-1-\lambda) - 2(-4-\lambda)$$
$$= -\lambda^3 - 3\lambda^2 - 4\lambda - 2 = 0.$$

This has an integer root $\lambda = -1$. Division by $\lambda + 1$ yields

$$-\lambda^2 - 2\lambda - 2 = 0,$$

which has roots $-1 \pm i$. An eigenvector for -1 must satisfy

$$3x_1 + 2x_2 + 9x_3 = 0, \quad x_1 + 3x_3 = 0, \quad -x_1 - x_2 - 3x_3 = 0.$$

We can use the second equation to find $x_1 = -3x_3$, and using this in the remaining two equations yields $x_2 = 0$. Hence the eigenvector is

$$\begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}.$$

An eigenvector for $-1 - i$ must satisfy

$$(3+i)x_1 + 2x_2 + 9x_3 = 0, \quad x_1 + ix_2 + 3x_3 = 0, \quad -x_1 - x_2 - (3-i)x_3 = 0.$$

We add the last two equations to find

$$(1-i)x_2 - ix_3 = 0,$$

leading to $x_3 = (-1-i)x_2$. Using this in the second equation, we find

$$-x_1 + (3+2i)x_2 = 0.$$

Hence $x_1 = (3+2i)x_2$. It can be checked that then the first equation is also satisfied. The eigenvector is therefore

$$\begin{pmatrix} 3+2i \\ 1 \\ -1-i \end{pmatrix}.$$

(You get the solution in the back of the book if you multiply this by $-1+i$.) The eigenvector for $-1+i$ is the complex conjugate,

$$\begin{pmatrix} 3-2i \\ 1 \\ -1+i \end{pmatrix}.$$

11. A complex solution is

$$e^{4t}(\cos 2t + i\sin 2t)\begin{pmatrix} 4 \\ -1+i \end{pmatrix} = e^{4t}\begin{pmatrix} 4\cos 2t + 4i\sin 2t \\ -\cos 2t - \sin 2t + i(\cos 2t - \sin 2t) \end{pmatrix}.$$

The real part is

$$e^{4t}\begin{pmatrix} 4\cos 2t \\ -\cos 2t - \sin 2t \end{pmatrix},$$

and the imaginary part is

$$e^{4t}\begin{pmatrix} 4\sin 2t \\ \cos 2t - \sin 2t \end{pmatrix}.$$

12. A complex solution is

$$(\cos t + i\sin t)\begin{pmatrix} -2+i \\ 5 \end{pmatrix} = \begin{pmatrix} -2\cos t - \sin t + i(\cos t - 2\sin t) \\ 5\cos t + 5i\sin t \end{pmatrix}.$$

The real part is

$$\begin{pmatrix} -2\cos t - \sin t \\ 5\cos t \end{pmatrix},$$

and the imaginary part is

$$\begin{pmatrix} \cos t - 2\sin t \\ 5\sin t \end{pmatrix}.$$

Solutions 4.7

3. a) The characteristic equation is

$$(-2-\lambda)^2 = 0,$$

i.e. we have an algebraically double eigenvalue -2. For an eigenvector, we find

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

i.e. $x_2 = 0$. Hence we have only one linearly independent eigenvector, namely

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

b) The equation for a generalized eigenvector is

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

i.e. $x_2 = 1$. We can, for instance, choose the vector

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

We now have the fundamental set

$$\mathbf{y}_1 = e^{-2t} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mathbf{y}_2 = e^{-2t} \begin{pmatrix} t \\ 1 \end{pmatrix},$$

and the fundamental matrix

$$\Psi = e^{-2t} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.$$

c) The general solution is

$$\mathbf{y} = c_1 \mathbf{y}_1 + c_2 \mathbf{y}_2.$$

For $t = 0$, we find

$$\mathbf{y}(0) = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

4. a) The characteristic equation is

$$(5 - \lambda)(1 - \lambda) + 4 = \lambda^2 - 6\lambda + 9 = (\lambda - 3)^2 = 0,$$

i.e. we have an algebraically double eigenvalue 3. For an eigenvector, we find

$$\begin{pmatrix} -2 & -1 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

i.e. $-2x_1 - x_2 = 0$. Hence we have only one linearly independent eigenvector, namely

$$\begin{pmatrix} -1 \\ 2 \end{pmatrix}.$$

b) The equation for a generalized eigenvector is

$$\begin{pmatrix} -2 & -1 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix},$$

i.e. $-2x_1 - x_2 = -1$. We can, for instance, choose the vector

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

We now have the fundamental set

$$\mathbf{y}_1 = e^{3t}\begin{pmatrix} -1 \\ 2 \end{pmatrix}, \quad \mathbf{y}_2 = e^{3t}\begin{pmatrix} -t+1 \\ 2t-1 \end{pmatrix},$$

and the fundamental matrix

$$\Psi = e^{3t}\begin{pmatrix} -1 & -t+1 \\ 2 & 2t-1 \end{pmatrix}.$$

c) The general solution is

$$\mathbf{y} = c_1\mathbf{y}_1 + c_2\mathbf{y}_2.$$

For $t = 0$, we find

$$\mathbf{y}(0) = \begin{pmatrix} -c_1 + c_2 \\ 2c_1 - c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

The solution is $c_1 = 2$, $c_2 = 3$.

16. a) If we set $z_1 = y$, $z_2 = y'$, we end up with the system

$$z_1' = z_2,$$

$$z_2' = -\alpha^2 z_1 + 2\alpha z_2.$$

b) The characteristic equation is

$$\begin{vmatrix} -\lambda & 1 \\ -\alpha^2 & 2\alpha - \lambda \end{vmatrix} = \lambda^2 - 2\alpha\lambda + \alpha^2,$$

i.e. we have a double eigenvalue α. For the eigenvector, we find

$$\begin{pmatrix} -\alpha & 1 \\ -\alpha^2 & \alpha \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

i.e. $z_2 = \alpha z_1$. This yields the eigenvector

$$\begin{pmatrix} 1 \\ \alpha \end{pmatrix}.$$

A generalized eigenvector must satisfy

$$\begin{pmatrix} -\alpha & 1 \\ -\alpha^2 & \alpha \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 1 \\ \alpha \end{pmatrix},$$

i.e. $-\alpha z_1 + z_2 = 1$. For instance, we can take the vector

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

c) The general solution is now

$$\mathbf{z} = c_1 e^{\alpha t} \begin{pmatrix} 1 \\ \alpha \end{pmatrix} + c_2 e^{\alpha t} \begin{pmatrix} t \\ 1 + \alpha t \end{pmatrix}.$$

Going back to the scalar variable, this is

$$y = c_1 e^{\alpha t} + c_2 t e^{\alpha t},$$

$$y' = c_1 \alpha e^{\alpha t} + c_2 (1 + \alpha t) e^{\alpha t}.$$

Solutions 4.8

2. a) The eigenvalues are $\pm i$ with eigenvectors $(1, i)$ and $(1, -i)$. The complementary solutions is

$$c_1 \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + c_2 \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}.$$

b) We get

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{a} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

which leads to

$$\mathbf{a} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

c)

$$\mathbf{y} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} + c_1 \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + c_2 \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}.$$

d) The initial conditions yield

$$2 + c_1 = 1, \quad -1 + c_2 = 1,$$

hence $c_1 = -1$, $c_2 = 2$.
5. a) The eigenvalues are ± 1, with eigenvectors $(1,1)$ and $(1,-1)$. The complementary solution is

$$c_1 e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

b) We find

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{a} + \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

$$\mathbf{a} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{b} + \begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$

The solution is

$$\mathbf{a} = \begin{pmatrix} 0 \\ -1 \end{pmatrix},$$

$$\mathbf{b} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

c)

$$\mathbf{y} = \begin{pmatrix} 0 \\ -t \end{pmatrix} + c_1 e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

d) The initial conditions lead to

$$c_1 + c_2 = 2, \quad c_1 - c_2 = -1.$$

This has the solution $c_1 = 1/2$, $c_2 = 3/2$.

Solutions 6.2

4. We want

$$xy - y + x - 1 = 0, \quad xy - 2y = 0.$$

The second equation can be factored as $(x-2)y = 0$, so we must have $x = 2$ or $y = 0$. For $x = 2$, the first equation yields

$$2y - y + 1 = 0$$

or $y = -1$. For $y = 0$, the first equation yields $x = 1$. Hence the equilibrium points are $(2,-1)$ and $(1,0)$.

7. We want

$$x^2 + y^2 - 8 = 0, \quad x^2 - y^2 = 0.$$

This is a linear system for x^2 and y^2 with solution $x^2 = y^2 = 4$. This results in the four equilibrium points $(-2, -2)$, $(-2, 2)$, $(2, -2)$ and $(2, 2)$.
16. For the system to have the given equilibrium points, we must have $\beta = \delta = 0$ and $2 + 2\alpha = \gamma - 6 = 0$, i.e. $\alpha = -1$, $\gamma = 6$. Hence the system is

$$x' = x - xy, \quad y' = 6y - 3xy.$$

The point $(-2, -2)$ is not an equilibrium point of this system.
24. The eigenvalues for the two given eigenvectors are -8 and -10, both negative. This corresponds to direction field C.
25. The eigenvalues for the two given eigenvectors are -4 and 2. Hence the direction must be inward on the first diagonal, outward on the second. This corresponds to direction field B.
26. The eigenvalues for the two given eigenvectors are 2 and -10. Hence the direction must be outward on the first diagonal, inward on the second. This corresponds to direction field D.
27. The eigenvalues for the two given eigenvectors are 6 and 2, both positive. This corresponds to direction field A.

Solutions 6.4

1. a) The ellipse

$$\frac{x^2}{4} + y^2 = C$$

has minor axis of length $\sqrt{C}$ and major axis of length $2\sqrt{C}$. Therefore, all solutions starting in a circle of radius δ never leave a circle of radius 2δ. We can set $\delta = \epsilon/2$.
b) The origin is not asymptotically stable. Solutions keep circling around the ellipses and do not get closer to the origin as time goes to infinity.
2. The system is unstable. If we pick $C < \delta^2$, then the hyperbola includes points within distance δ of the origin, but other points on the hyperbola will be as far from the origin as we want (i.e. more than any ϵ).
9. The eigenvalues are 4 and 2, so the origin is unstable.
10. The eigenvalues are $-2 \pm 3i$, so the origin is asymptotically stable.

Solutions 6.5

1. a) The equations for equilibrium points are $x^2+y^2=32$, $y-x=0$, which has the solutions $x=y=\pm 4$.
b) At the point $(4,4)$, we have

$$\mathbf{A} = \begin{pmatrix} 8 & 8 \\ -1 & 1 \end{pmatrix},$$

which has eigenvalues $(9 \pm \sqrt{17})/2$. The point is unstable.
At the point $(-4,-4)$, we have

$$\mathbf{A} = \begin{pmatrix} -8 & -8 \\ -1 & 1 \end{pmatrix},$$

with eigenvalues $(-7 \pm \sqrt{113})/2$. Since $\sqrt{113} > 7$, this point is also unstable.
6. a) The first equation yields $y = x$ or $y = -1$. If we choose $y = x$, the second equation leads to $x = y = -2$ or $x = y = 4$. If we choose $y = -1$, the second equation yields $x = -2$. Hence the equilibrium points are $(-2,-2)$, $(4,4)$ and $(-2,-1)$.
b) We write the system in the form

$$x' = xy - y^2 + x - y,$$

$$y' = xy + 2y - 4x - 8.$$

The matrix of the linearization at the point (x_e, y_e) is

$$\begin{pmatrix} y_e + 1 & x_e - 2y_e - 1 \\ y_e - 4 & x_e + 2 \end{pmatrix}.$$

For the point $(-2,-2)$, we find the matrix

$$\begin{pmatrix} -1 & 1 \\ -6 & 0 \end{pmatrix},$$

which has eigenvalues $(-1 \pm i\sqrt{23})/2$. This point is stable.
For the point $(4,4)$, we find the matrix

$$\begin{pmatrix} 5 & -5 \\ 0 & 6 \end{pmatrix},$$

which has eigenvalues 5 and 6. This point is unstable.
For the point $(-2,-1)$, we find the matrix

$$\begin{pmatrix} 0 & -1 \\ -5 & 0 \end{pmatrix},$$

which has eigenvalues $\pm\sqrt{5}$. This point is unstable.