

# Flexible Galerkin Finite Element Methods

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## Abstract

We present a new class of flexible Galerkin methods that allow for nonuniform continuity levels across element boundaries.

*Keywords:* Finite elements; Discontinuous Galerkin; Flexible Galerkin

## 1 Introduction

The flexible Galerkin (FG) finite element method for solving partial differential equations is a hybrid of the standard continuous Galerkin (CG) method and the discontinuous Galerkin (DG) method. A major overhead associated with DG methods comes from the large number of degrees of freedom because entities on inter-element boundaries are not shared. In an attempt to reduce the computational cost, we prescribe for each element the continuity levels on mesh entities shared by neighboring elements. The FG method may be used to implement *hp* adaptive continuous/discontinuous coupling methods [2] in an efficient manner.

## 2 The Flexible Galerkin method

### *2.1. Discretization*

We illustrate the FG method on the linear hyperbolic problem

$$\boldsymbol{\alpha} \cdot \nabla u = f(x, y) \quad \text{in } \Omega = [0, 1] \times [0, 1], \quad \alpha_i > 0, \quad i = 1, 2, \quad (2.1)$$

with inflow boundary conditions.

We subdivide  $\Omega$  into  $N = n \times n$  square elements and start the integration from the element  $\Delta = [0, h]^2$ ,  $h = 1/n$  whose inflow boundary is on the inflow boundary of the domain. We continue the integration on elements whose inflow boundary is either an outflow boundary of  $\Delta$  or is on the inflow boundary of  $\Omega$ . Let  $\Gamma$  ( $\Gamma_{in}$ ) denote the boundary (inflow boundary) of  $\Delta$ .

We construct the space  $\mathcal{V}_p = \{ V \mid V = \sum_{k=0}^p \sum_{i=0}^k c_i^k x^i y^{k-i} + \sum_{i=1}^p c_i^{p+1} x^i y^{p+1-i} \}$  using the hierarchical shape functions of Szabó and Babuška [3]. The approximation  $U(x, y) \in \mathcal{V}_p$ ,  $p \geq 1$ , is spanned by vertex modes, edge modes and interior modes. Only vertex and edge modes are allowed to be discontinuous.

In order to obtain an efficient  $hp$  adaptive flexible Galerkin method, the representation is hierarchical with element having pointers to their bounding edges which, in turn, have pointers to their bounding vertices. Furthermore, each vertex (edge) has a degree  $p$ , a continuity level  $0 \leq c \leq p$  and pointers to elements sharing the vertex (edge). When solving (2.1) on each element, we use  $c = 0$  on the outflow mesh entities. Data for the discontinuous solution component and interior shape functions is stored with the element. Data for the continuous edge (vertex) solution are stored with the corresponding edge (vertex). For more details on the FG method consult [1].

## 2.2 Weak formulation

The finite element solution can be written as  $U(x, y) = U_c(\xi, \eta) + U_d(\xi, \eta)$ , where  $U_c$  ( $U_d$ ) denote the continuous (discontinuous) component of  $U$ . The FG method consists of determining  $U \in \mathcal{V}_p$  such that

$$\oint_{\Gamma_{in}} \boldsymbol{\alpha} \cdot \mathbf{n} [U - U^-] V \, ds - \iint_{\Delta} (\boldsymbol{\alpha} \cdot \nabla U - f) V \, dx \, dy = 0, \quad \forall V \in \mathcal{V}_p^o,$$

where  $U^-(x, y) = \lim_{s \rightarrow 0^+} U((x, y) + s\mathbf{n})$ ,  $\mathcal{V}_p^0 = \{U \in \mathcal{V}_p \mid U_c|_\Gamma = 0\}$  and  $\mathbf{n}$  is the outward unit normal.

The number of degrees of freedom for the FG method with a finite element space  $\mathcal{V}_p$  and a uniform continuity level  $c$  can be written as  $N(p, c, d) = 4p + (p - 1)(p - 2)/2 - (1 - \delta_{0c})(d - c + 1)$ , where  $d$  is the number of inflow edges and  $\delta_{ij}$  is the *Kronecker delta*.

### 2.3. A numerical example

We solve (2.1) with  $\boldsymbol{\alpha} = [1, 2]^t$  and exact solution  $u(x, y) = e^{x^2 - y^2}$  using  $p = 1$  to 4,  $c = 0$  to  $p$  on uniform meshes having 25, 100, 225, 400, 625, and 900. In Figure 1 we plot the FG convergence rates in the  $\mathcal{L}^2$  norm versus  $h$  for  $c = 0$  to  $p$ . When  $c = 0$  we recover the optimal DG  $O(h^{p+1})$  convergence rate and when  $c = p$  we obtain the CG  $O(h^p)$  convergence rate. When  $p \geq 2$  and  $c = 1$ , the FG convergence rate is approximately  $O(h^{p+\frac{1}{2}})$ . If  $p \geq 3$  and  $2 \leq c < p$ , the FG convergence rate approaches that of the CG method.

## 3 Conclusions

We developed a new flexible Galerkin method that leads to efficient  $hp$  adaptive CG/DG methods with variable continuity level across element boundaries. The convergence rate is  $O(h^{p+1})$ , for  $c = 0$ ,  $O(h^{p+1/2})$  for  $c = 1$  and  $O(h^p)$  for  $c \geq 2$ . Rigorous error analysis for the FG finite element method is yet to be performed.

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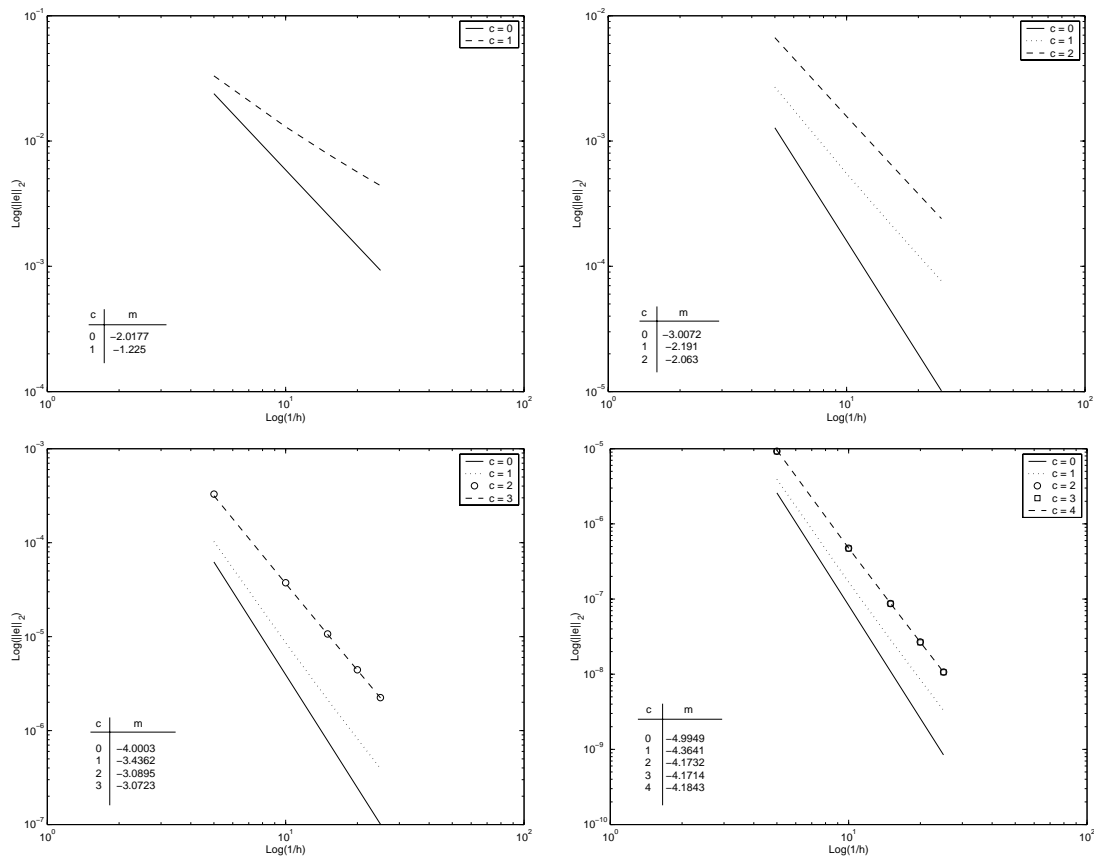


Figure 1: Rates of  $h$ -convergence for  $\|e\|_{\mathcal{L}_2(\Omega)}$ , denoted by  $m$ , for  $p = 1$  to 4 (upper left to lower right) with  $c = 0$  to  $p$ .