Modeling Population Growth

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Growth of bacteria

- Population starts with 50 bacteria
- Each bacteria splits into 2 every hour
- How many are left after 3 hours?
TIME \( t \)  

POPULATION

\[
\begin{align*}
\text{start:} & \quad t = 0 \\
\text{after 1 hour:} & \quad t = 1 \\
\text{after 2 hours:} & \quad t = 2 \\
\text{after 3 hours:} & \quad t = 3
\end{align*}
\]

\[
\begin{align*}
50 & \\
50 \cdot 2 = 100 & \\
50 \cdot 2 \cdot 2 = 50 \cdot 2^2 = 200 & \\
50 \cdot 2 \cdot 2 \cdot 2 = 50 \cdot 2^3 = 400 & \\
\end{align*}
\]

Look familiar?

after \( t \) hours:

\[
\begin{align*}
50 \cdot 2^t & \quad \text{number of splits} \\
\text{initial population} & \quad \text{split factor}
\end{align*}
\]
Can the population grow forever?

- Not in biological systems!
- What limits growth?
Modeling populations

The population in 1 hour depends on the population this hour:

$$\text{Pop (in 1 hour)} = \text{Pop (now)} + \text{births} - \text{deaths}$$
Example population model

\[ P(t+1) = \frac{P(t)}{P(t)+a} \quad a > 0 \]
Some definitions

**Iteration**: one application of the function on the right hand side

\[ P(t+1) = \frac{P(t)}{P(t)+a} \]
Some definitions

**Iterate:** size of the population after one iteration

\[ P(t+1) = \frac{P(t)}{P(t)+a} \]
More definitions

**Fixed point:** population size that does not change after one iteration

Look for where $P(t+1) = P(t)$
Can you find a fixed point?

**Fixed point**: population size that does not change after one iteration

\[ P = \frac{P}{P + a} \]
More definitions

**Accumulating point:** population size that occurs repeatedly after iterations

May not return to an accumulating point after one iteration
Can you find an accumulating point?

**Accumulating point:** population size that occurs repeatedly after iterations

\[ P(t+1) = \frac{P(t)}{P(t) + a} \quad a = 0.1 \]
Graphical solutions

![Graphical solutions](image-url)
Getting to the fixed points: Cobweb diagrams

What happens if we start at $x = 0.3$?
Getting to the fixed points: Cobweb diagrams

After one iteration
Getting to the fixed points: Cobweb diagrams

Use that value as a new starting condition
Getting to the fixed points: Cobweb diagrams

![Cobweb diagram showing iteration process and fixed points.](image-url)
Getting to the fixed points: Cobweb diagrams

Keep going until you reach a fixed point
Getting to the fixed points: Cobweb diagrams

What if we start at $x = 0.2$ or $x = 0.8$?
Getting to the fixed points: Cobweb digrams

Start at $x = 0.2$
Getting to the fixed points: Cobweb diagrams

Start at $x = 0.8$
Getting to the fixed points: Cobweb diagrams
Can we ever get to zero?
Now, imagine a fish tank

How many fish are there?
Only so many fish can fit before overcrowding becomes a problem!
We can model this!

- Let $x(t)$ be the number of fish in the tank this week
- Let $r$ be the amount of food you feed them each week
- Fish population grows slowly when there are more fish
- After one week:

$$x(t+1) = r \times x(t) \times (1 - x(t))$$
Does it matter how many fish start in the fish tank?
Does it matter how much you feed the fish?
Try cobwebbing solutions

[Handouts]
Let’s look at cobweb digrams
The diagram shows a logistic growth model with a bifurcation parameter $r = 0.50000$. The x-axis represents $x(t)$, and the y-axis represents $x(t+1)$. The graph illustrates the behavior of the system as $x(t)$ changes with increasing $t$. The bifurcation point is evident at $r = 0.5$, indicating a transition from a stable fixed point to a more complex dynamics.
$r = 0.50000$
$r = 1.50000$
$r = 2.50000$
$r = 2.90000$
$r = 3.10000$
\( r = 3.10000 \)
$r = 3.20000$
$r = 3.30000$
$r = 3.30000$
$r = 3.50000$

The graph shows a time series plot with the x-axis labeled as "iteration" and the y-axis labeled as $x(t)$. The data points are plotted for iterations from 0 to 50, showing a pattern that appears to oscillate.
$r = 3.55000$
$r = 3.55000$

The graph shows a time series plot of $x(t)$ over iterations. The parameter $r$ is set to 3.55000, which is a critical value for the period-doubling bifurcation in the logistic map.
$r = 3.56995$
$r = 3.56995$

The diagram shows a time series of $x(t)$ values with a period of 4 for the first 50 iterations, indicating a phenomenon with a period doubling bifurcation.
$r = 3.82840$
$r = 3.82840$
The diagram illustrates the logistic map with a parameter $r = 3.82845$. The map is defined as $x_{t+1} = r x_t (1 - x_t)$, where $x_t$ represents the population at time $t$, and $r$ is the growth rate. The graph shows the bifurcation diagram for this map, with iterations of $x_t$ plotted against $x_{t+1}$, illustrating the transition from a stable fixed point to periodic behavior and eventually chaos as $r$ increases.
$r = 3.82845$
$r = 3.86000$
r = 3.86000
$r = 4.00000$
$r = 4.00000$

The graph shows a time series with the x-axis representing iteration and the y-axis representing $x(t)$. The data points are plotted at each iteration, showing a periodic pattern as $r$ increases to a critical value.
Can we summarize this?
Look at points in the trajectory after many iterations

\[ r = 4.00000 \]
How dynamics of $x$ change with $r$
[Movie of bifurcation diagram]
Regions of chaos

• No fixed points

• No accumulating points

• Trajectory never repeats

• Sensitive dependence on starting point
Sensitive dependence on starting point
Chaos can appear even in simple models!
Does it matter how many fish start in the fish tank?

No!
Does it matter how much you feed the fish?

Yes!
THANK YOU!