Volumes: General Solids

We continue with our work on the volumes of solids. Today, we will work with more general volumes, where the cross-section is not necessarily a circle.

General Volumes

Recall that in general, we find the volume of a solid by finding the area of a general cross-sectional slice along some axis. We’ll call the axis “$h$” for “height”, and then the volume of a solid is given by the following, where $A(h)$ is the cross-sectional area at height $h$:

\[
\text{Volume} = \int_a^b A(h) \, dh
\]

(The numbers $a$ and $b$ mark the “top” and “bottom” of the solid along the $h$ axis.)

In the case of solids of revolution, all the cross sections are circles, so we just ended up with the simpler formula

\[
V = \pi \int_a^b r(h)^2 \, dh
\]

where $r(h)$ is the radius of the cross-sectional slice at height $h$. But if the cross-sections are not circles, we cannot use this formula!

Example: Find the volume of a cylinder that has an elliptical base with semi-major axis 5, semi-minor axis 4, and height 7.

The picture of such a solid is shown below:

Here, each cross-sectional slice across the $z$-axis is an ellipse. The formula for the area of an ellipse is $\pi ab$, where $a$ and $b$ are the semi-major and semi-minor axes.

Let’s set this up with the height along the $z$-axis. So in this case, since every slice is the same, we have $A(z) = \text{___________}$. Then we simply integrate from 0 to 7:
Examples for Practice

Try the following in your group. For each problem, try to draw diagrams to help you figure out what the cross-sections look like and what the area of each cross-section is.

If your group finishes setting up both problems, evaluate the integrals to find the indicated volumes.

Example 1:  The Great Pyramid of Egypt is actually the Great Pyramid of Giza. It is one of the Ancient Wonders of the World built around 2550 BC as a tomb for the Pharaohs. It is the only wonder of the ancient world that is still standing!

Its measurements are:

480 feet high. There are a couple different heights mentioned but this was given the most often.

Its base is a square 756 feet on a side.

Our goal will be to find the volume of this pyramid.

We will therefore need to find the cross-sectional area at any height $h$. Since the cross-sections are squares, we need to know the length of the side of the pyramid at each height $h$.

So do the following:

First: Use similar triangles to show that the length of the side $s$ of the square horizontal cross section of the pyramid at any height $h$ between 0 and 480 feet is given by the formula

$$s(h) = \frac{756}{480} (480 - h).$$

(Hint: Start by sketching a vertical slice of the pyramid, and label everything you can.)

Second: Find a general formula for the area of the square cross section at any height $h$.

Third: Set up an integral to find the volume of the Great Pyramid in millions of cubic feet.

Then evaluate:
If you finish this problem, compute the volume of the Great Pyramid using your answer from part (c). Compare your answer to the geometric formula for the volume of a square pyramid of height $h$ with base length $s$:

$$V = \frac{1}{3} s^2 h = \frac{1}{3} (756)^2 480 = 91,445,760 \text{ ft}^3$$

The **Hemispherical Dome** is half of a sphere. It was a shape likely known to the Assyrians, defined by the Greek theoretical mathematicians and standardized by Roman Builders. (Wikipedia) The Pantheon in Rome, Italy was first built as a temple to the gods during the reign of Augustus and rebuilt about 126 AD.

**Example 2:** Set up and solve an integral to find the volume of the Pantheon Dome whose diameter is 142 ft. Hints:

1) Draw the picture. Sketch a side view with the cut center of the dome at (0, 0) and the $z$-axis pointing up to the top of the dome.

2) There are several different circles involved here. It will be useful for you to remember that the equation which describes the $x$ and $z$ coordinates of a circle of radius $R$ in the $x$-$z$ plane is $x^2 + z^2 = R^2$. 
When you finish this problem, compare the answer for the volume of the Pantheon Dome to the geometric formula for the volume of a hemisphere of radius $R$: $V = \frac{2}{3} \pi R^3$ =

Example 3:
Using calculus find the volume of a square based pyramid of height 40 ft and length of each side of the base 80 ft.

Example 4:
Using calculus find the volume of a hemisphere of radius 10 ft.

**Summary**

Today, we have

- Discussed more about finding volumes, including general solids which may or may not have circular cross sections.
- Practiced setting up and solving volume problems.