Improper Integrals

Today we consider definite integrals with infinite limits.

**Introduction**

Let’s consider area under the curve $y = e^{-x}$ for $x > 0$. It is easy to determine the area over any finite extent. For example:

- from $x = 0$ to $x = 1$, the area is $\int_0^1 e^{-x}dx =$
- from $x = 0$ to $x = 5$, we get $\int_0^5 e^{-x}dx =$
- and from $x = 0$ to $x = 10$, we get $\int_0^{10} e^{-x}dx =$

Hmm… in fact, the area from $x = 0$ to $x = b$ is given by $\int_0^b e^{-x}dx =$

Since $e^b$ becomes small as $b$ becomes larger, this answer gets closer and closer to 1 as we move further and further to the right. So perhaps all the area under the curve to the right of 0 turns out to be 1.

In fact, if we look at the curve $e^{-x}$, we see that the area under each successive section of the curve get smaller and smaller, and so add less and less total area. We can imagine that the increase in the integral from adding each new integral gets smaller and smaller, so that the growth eventually “sputters out” and never gets bigger than one.

It may seem strange to think of an integral which “goes on forever” like this, but there are some applications in which this is reasonable. It might describe the gradual absorption of a drug into the body, or the gradual assimilation of a species into a new region. It is also used in probability, which we will study later in the semester.
The Improper Integral

With the idea we started above, we will give meaning to integrals with limits that go to \( \pm \infty \) as follows:

\[
\int_{a}^{\infty} f(x) \, dx \quad \text{will mean } \quad \lim_{b \to \infty} \int_{a}^{b} f(x) \, dx,
\]

and

\[
\int_{-\infty}^{b} f(x) \, dx \quad \text{will mean } \quad \lim_{a \to -\infty} \int_{a}^{b} f(x) \, dx.
\]

What this means for us is that when we see a so-called improper integral (that is, an integral with an infinite limit), we must first find an expression for the integral is over any finite extent, then take the limit as the upper limit goes to infinity (or lower limit goes to negative infinity).

**Example:** Find \( \int_{1}^{\infty} \frac{1}{t^{2}} \, dt \).

Start by finding \( \int_{1}^{b} \frac{1}{t^{2}} \, dt = ? \). Now we need to determine what happens as \( b \) goes to \( \infty \). If you don’t remember how to deal with limits, you could try calculating a few values to get some idea:

- If \( b = 1 \),
- If \( b = 10 \),
- If \( b = 100 \),

We begin to see a pattern; as \( b \) becomes large, \( 1/b \) becomes small, and we get closer and closer to \( ? \) for the value of the integral. Therefore, we say that \( \int_{1}^{\infty} \frac{1}{t^{2}} \, dt = ? \).

**Example:** Calculate \( \int_{-\infty}^{2} e^{3x} \, dx \).

It is entirely possible that the limit does not exist at all. (In fact, this would be true of most integrals; often the limit goes to \( \infty \) or \( -\infty \).)

**Example:** Find \( \int_{0}^{\infty} x^{2} \, dx \).
If the limit does not exist, people will say that the integral *diverges*.
Sometimes if the limit does not exist we may also say $\int_0^\infty f(x)\,dx = \infty$,
which means both that the integral doesn’t exist and says explicitly that it
doesn’t exist because the value goes to infinity as we move to the right.

**Both Limits Infinite**

We also occasionally see integrals in which both limits are infinite, such as

$$\int_{-\infty}^\infty f(x)\,dx$$

These can actually be fairly tricky, and the only way to be sure these are exactly right is
to split them into two integrals, and calculate each separately. For example:

However, if we are sure that an answer exists, we can also find it by using a formula like

$$\int_{-\infty}^\infty f(x)\,dx = \lim_{b \to \infty} \int_{-b}^b f(x)\,dx$$

and taking both limits at once.

**Summary**

Today we have

- Given meaning to improper integrals (of the form $\int_a^\infty f(x)\,dx$, $\int_{-\infty}^b f(x)\,dx$, and
$\int_{-\infty}^\infty f(x)\,dx$) by making these the limit of a definite integral with variable upper or
lower limit.
- Found the value of such improper integrals.
- Had a caution regarding improper integrals with both limits infinite ($\int_{-\infty}^\infty f(x)\,dx$),
since these are somewhat more complicated. We must first break these apart into
two improper integrals, and find the value of each separately in order to evaluate
these integrals.