Antiderivatives

Today we begin a study of antiderivatives. Antiderivatives will be useful for evaluating integrals using the Fundamental Theorem, and for solving some initial value problems.

**Definition**

We say that $F$ is an antiderivative for the function $f$ if $F'(x) = f(x)$.

**Example:** We see that
- $x^2$ is an antiderivative for $2x$, because $\frac{d}{dx}(x^2) = 2x$.
- What is an antiderivative for $x^2$? We might guess $x^3$ at first, but we see that $\frac{d}{dx}(x^3) = 3x^2$, so this is not correct. If we multiply by $\frac{1}{3}$ first to cancel the 3, we get a new guess, $\frac{1}{3}x^3$. We see that $\frac{1}{3}x^3$ is an antiderivative for $x^2$, because $\frac{d}{dx}\left(\frac{1}{3}x^3\right) = x^2$.

We found these by guessing an antiderivative, and then checking by taking the derivative. We will learn other ways to find antiderivatives, but you will always be able to check your answer by taking the derivative.

**Exercises**

Find an antiderivative to each of the following functions. (Guess, and then be sure to check your answer. If your guess wasn’t right, change your guess and try again.)

- $f(x) = x$
  Antiderivative:

- $f(x) = x^3$
  Antiderivative:

- $f(x) = 3x^7$
  Antiderivative:

- $f(x) = \sin(x)$
  Antiderivative:

- $f(x) = \cos(x)$
  Antiderivative:

- $f(x) = e^x$
  Antiderivative:

- $f(x) = \frac{1}{x}$
  Antiderivative:

- $f(x) = 7$
  Antiderivative:

- $f(x) = x + e^x$
  Antiderivative:
### Antiderivative Rules

Mathematics includes a lot of pattern matching. Based on some of the problems we saw in the exercises, we can start to establish some patterns. The following rules will help us find antiderivatives:

<table>
<thead>
<tr>
<th>If $f$ is</th>
<th>then an antiderivative is</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x^n$</td>
<td>$\frac{1}{n+1} x^{n+1}$ except if $n = -1$</td>
</tr>
<tr>
<td>$1$</td>
<td>$x$ (if the variable is $x$)</td>
</tr>
<tr>
<td>$\cos(x)$</td>
<td>$\sin(x)$</td>
</tr>
<tr>
<td>$\cos(kx)$</td>
<td>$\frac{1}{k} \sin(kx)$</td>
</tr>
<tr>
<td>$\sin(x)$</td>
<td>$-\cos(x)$</td>
</tr>
<tr>
<td>$\sin(kx)$</td>
<td>$-\frac{1}{k} \cos(kx)$</td>
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<tr>
<td>$e^x$</td>
<td>$e^x$</td>
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<tr>
<td>$e^{kx}$</td>
<td>$\frac{1}{k} e^{kx}$</td>
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<tr>
<td>$a^x$</td>
<td>$\frac{1}{\ln a} a^x$</td>
</tr>
<tr>
<td>$\frac{1}{x}$</td>
<td>$\ln</td>
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</tbody>
</table>

We also have an easy rule for antiderivatives when we have sums or constant multiples: If $F$ is an antiderivative for $f$, and $G$ is an antiderivative for $g$, and $b$ and $c$ are constants, then

$$b \, F(x) + c \, G(x) \text{ is an antiderivative for } b \, f(x) + c \, g(x).$$

In other words, in taking antiderivatives, we can work through sums and constant multiples just like in taking derivatives. (This really contains two rules: a sum/difference rule and a constant multiple rule.)

**Example:** An antiderivative of $3x + 7 - 3e^x$ is

**Example:** To find an antiderivative for $\frac{3}{x} + \frac{3}{x^2}$, rewrite it as $\frac{1}{x} + 3x^{-2}$. Then finding an antiderivative is easy
We see that the so-called **power rule** for functions of the form \( x^n \) is rather versatile. Here’s another example:

**Example:** To find an antiderivative for \( \sqrt{x} \), first rewrite the function as \( x^{1/2} \). Then the power rule says that an antiderivative is

**The Indefinite Integral**

Because the fundamental theorem says that antiderivatives are useful for evaluating definite integrals, we adopt a notation which looks like an integral to denote finding antiderivatives.

We say that the **indefinite integral** of a function \( f \), written as

\[
\int f(x) \, dx
\]

represents all possible antiderivatives for the function \( f \). (Note that the indefinite integral looks like the definite integral but with no limits.)

So for example, if we had \( \int x^2 \, dx \), we would need to find all antiderivatives of \( x^2 \). We know from our power rule that one antiderivative is \( \frac{1}{3}x^3 \). But since the derivative of any constant is zero, \( \frac{1}{3}x^3 + 7 \) and \( \frac{1}{3}x^3 - \pi \) are also antiderivatives of \( x^2 \). In fact, the set of all antiderivatives of \( x^2 \) is

\[
\int x^2 \, dx =
\]

where \( C \) is any constant.

In general, to find all antiderivatives of a function \( f \), just find one antiderivative \( F \). Then the set of all antiderivatives is just

\[
\int f(x) \, dx = F(x) + C
\]

*An Aside:* We certainly haven’t proved that all possible antiderivatives are only different by a constant, but it makes some intuitive sense. Since the derivative of our function is known, we know where it increases and decreases, and by exactly how much. The only other way antiderivatives could be different is to slide up and down on the \( y \)-axis, which corresponds to adding a constant.

We could rewrite some of our rules in our new notation. For example, the power rule is

\[
\int x^n \, dx =
\]

while the constant multiple rule and sum rule look like this:

\[
\int k f(x) \, dx = \quad , \quad k \text{ a constant}, \quad \text{and}
\]

\[
\int [f(x) + g(x)] \, dx =
\]
Example: \[ \int (3\sin(x) + 2)\,dx = \]

Example: \[ \int \left( \frac{2}{\sqrt{q}} + e^q - 6 \right)\,dq = \]

**Using Algebra To Find Antiderivatives**

How could we find \[ \int (x-1)(x+1)\,dx \] ? You could do this by first algebraically rearranging the integrand:

\[ \int (x-1)(x+1)\,dx = \int (x^2 - 1)\,dx = \]

There are many cases in which algebra will help us put the integrand into a form for which we have an antiderivative formula. Anytime we have products of polynomials like we did above, we can multiply first and then take the antiderivative:

Example: \[ \int (3x + 1)^2\,dx = \]

**WARNING:** We have no rules for integrals of products, so multiplying out the integrand is pretty much the only way we can handle a product for now.

We can also sometimes simplify quotients using algebra:

Example: \[ \int \frac{x + x^2}{x}\,dx = \]

Example: \[ \int \frac{\sqrt{t}}{t^2}\,dt = \]

**WARNING:** Only multiplication cancels division! **Don’t** try simplifications like the following:

***WRONG*** \[ \frac{x}{x^2 + 1} = \frac{1}{x + 1} = \frac{1}{x} \] ***WRONG***

Both steps above are wrong! We cannot cancel the \( x \) with the \( x^2 \); we can only cancel common factors on the top and bottom. And please remember that \( \frac{1}{a + b} \neq \frac{1}{a} + \frac{1}{b} \).

For now, we do not know how to calculate \[ \int \frac{x}{x^2 + 1}\,dx \]. (But soon we will!)
**What About C?**

We know that if \( F \) is an antiderivative for \( f \), then \( F(x) + C \) is also. If we want to be able to choose a specific function \( F \) to use for the antiderivative, we need more information than just \( f \). We need what is called an *initial condition*, which is a value for \( F \). We first find the indefinite integral, then solve algebraically for \( C \).

**Example:** Find a function \( F(x) \) such that \( F'(x) = \sin(x) + 3x \) and \( F(0) = 2 \).

Start by finding the antiderivative:

So this is our candidate for \( F \). Now determine \( C \), based on the fact that \( F(0) = 2 \):

**Summary**

Today, we have

- Defined what an antiderivative is.
- Developed formulas for antiderivatives. In particular, we can find antiderivatives of any power function, exponential functions, sine and cosine, and combinations of these.
- Determined that we can check any antiderivative by taking the derivative.
- Defined the indefinite integral \( \int f(x) \, dx \) as the set of all antiderivatives of the function \( f \). We can find all antiderivatives of a function by finding one antiderivative and then adding an arbitrary constant.
- Used algebraic simplification of products or quotients to evaluate integrals.
- Found the specific choice of \( C \) for an antiderivative from an initial condition.