Integral Properties; Average Value

Today we discuss algebraic properties of integrals, and then consider finding the average value of a function on an interval.

Three Properties

The three properties we will discuss are given on p. 40 of the text. We assume that \( a, b, \) and \( c \) are any numbers, and that \( f \) and \( g \) are continuous functions.

Additive Property:

This property is easy to understand in terms of areas. If we want to find the area under the curve from \( a \) to \( c \), we can do this by first finding the integral from \( a \) to \( b \) then adding it to the integral from \( b \) to \( c \):

Sums and Differences:

This says we can split up sums or differences of two different functions. This makes good sense, since the integral itself is like a sum, and we can always rearrange addition.

Constant Multiples:

This property says we can take any constant multiple \( c \) out in front of the integral sign. The property makes sense if we think about integrals as sums. After all, we can always factor out constant multiples from a sum.

It is important to note that \( c \) must be a constant, like 7 or –34.8. We cannot bring functions outside the integral.
Two Conventions

In order for our rules to work for all possible choices for \(a, b,\) and \(c,\) we also need to agree to two not very surprising conventions:

- The integral of any function with upper and lower limits equal is just zero. We discussed this previously, saying that there was no space for there to be any area above or below the axis.
- This allows us to give meaning to “backwards” integrals. It makes some sense that if going forward means adding up, then going backwards must mean subtracting.

We will not use these conventions very often, but it’s good to agree on what things like \(\int_1^1 f(x) \, dx\) or \(\int_2^1 f(x) \, dx\) mean when we run across them.

Five Examples

Now we try some calculations using these properties.

Example: Suppose the following four integrals are known for the functions \(f\) and \(g:\)
\[
\int_0^2 f(x) \, dx = 4 \quad \int_0^2 g(x) \, dx = -1 \\
\int_0^1 f(x) \, dx = 1 \quad \int_0^1 g(x) \, dx = 3
\]

Use these facts to evaluate the following, if possible:
\[
\int_0^1 2f(x) + g(x) \, dx =
\]
\[
\int_0^1 g(x) \, dx =
\]
\[
\int_0^1 -5f(x) \, dx =
\]
\[
\int_0^1 f(x) \cdot g(x) \, dx =
\]
\[
\int_0^1 4f(x) − 3g(x) \, dx
\]
**Average Value of a Function**

We now turn to the definition of the average value of a function on an interval. To motivate the definition, we consider the following:

We know how to average a list of values. The average of $a_1, a_2, a_3, \ldots, a_n$ is just

$$\text{Average of } f \text{ on } [a, b]$$

But what then should we call the “average value” of a function $f$ on some interval $[a, b]$? One approach is to split the interval into a bunch of little subintervals of width $\Delta x$, pick a point $x_i$ in each subinterval, and then average the $f(x_i)$:

The approximation should get better as $n$ gets larger and the subintervals get smaller. This sounds a lot like an integral. The only thing we’re missing is a $\Delta x$ in the formula. Let’s use the fact that $\Delta x = (b - a)/n$ to write

Then as $n$ goes to infinity, the top becomes an integral, and the bottom just stays $b - a$, so we get

$$\text{Average value of } f \text{ on } [a, b] =$$

We will take the formula above as our definition of average value.
An Example and a Graphical Interpretation

We consider an example, and discuss a graphical interpretation of the average value.

**Example:** Find the average value of \( f(x) = x^2 \) on the interval \([-1, 1]\).

By definition, the average value should be

\[
\frac{1}{b-a} \int_a^b f(x) \, dx = \frac{1}{b-a} \int_{-1}^{1} (x^2) \, dx = \frac{1}{b-a} \int_{-1}^{1} x^2 \, dx
\]

We can compute the integral using Simpson’s rule, since the integrand is quadratic:

Finally, the average value is ______________.

In the graph sketched below, we can see a graphical interpretation for average value on an interval:

The curve \( y = x^2 \) and the line \( y = 1/3 \) are sketched on the graph. We can see there is as much area under the line as under the curve. The area under \( f \) (counting areas beneath the axis as negative of course) is the same as a rectangle with height the average value of the function on the interval.

Or, to say the same thing another way, if \( A \) is the average value on \([a, b]\), then

So the integral will be the same if instead of calculating \( f(x) \) at each point, we simply use the average value everywhere.
**Another Example**

We consider a few more examples, attempting to hone our intuition of what an average value looks like.

**Example:** Find the average value of $f(t) = 2^t$ on the interval $[0, 2]$.

The graph is shown below. Make a reasonable guess as to about where the average value will be before attempting any calculations:

![Graph of $f(t) = 2^t$](image)

We know the average value will be

Now we do not know a way to evaluate the integral exactly, so let’s make a moderately good estimate using Simpson’s rule. Although this is a smooth curve, it is an exponential function, and therefore grows much more quickly than parabolas. Let’s use $n = 4$ subintervals:

![Graph of $f(t) = 2^t$](image)

So our average value is $1/2$ times this, or about ______. How does this compare with your graphical estimate of the average?
Summary

Today, we have

- Given algebraic properties of integrals and used algebraic properties to find the value of integrals.

- Motivated and stated the formula for the average value of a function \( f \) on an interval \([a, b]\):

\[
\frac{1}{b-a} \int_{a}^{b} f(x) \, dx
\]

- Calculated average values.

- Estimated the average value from a graph.

- Interpreted the average value graphically: it is the height or \( y \)-value which would give the same integral as the function over the interval.