Differential Equations

We begin a brief study of differential equations, which are simply equations involving derivatives which we must solve for a function rather than a value. Differential equations are useful in modeling a wide variety of physical and biological phenomena.

What’s a Differential Equation?

When we solve an algebraic equation, we are looking for a number we can plug in for a variable that will make the equation true. For example, find the solutions to $x^2 - 4 = 0$.

A differential equation is an equation involving an unknown function and its derivative (or several derivatives). The solution to a differential equation is a function that makes the equation true.

Example: A very simple differential equation is $\frac{dy}{dx} = 2x$. A solution to this differential equation is $y = x^2$, because the derivative $dy/dx$ is in fact $2x$. (There are obviously other solutions to this differential equation.)

Example: Here’s a more typical differential equation: $y' = 3y$. Note that this one says that we are looking for a function $y$ whose derivative is three times $y$, or three times the original function. The independent variable is not specified in this equation.

One solution to $y' = 3y$ is $y = 2e^{3t}$, because

We chose to use $t$ for the independent variable, but we could have used anything else since it is not specified.

Verifying Solutions

As we saw in the above examples, it is very easy to verify that a function is the solution to a differential equation: we simply plug in our proposed function and see if the equation is true. We must remember however that one variable represents the function itself, not a variable to be differentiated or otherwise changed.
Example: Determine which of the following are solutions to \( y' = 3y \). (We already determined that \( y = 2e^{3t} \) was a solution.)

- \( y = e^{3t} \)
- \( y = e^{3t} + 1 \)
- \( y = 3t \)

Example: Confirm that \( y(t) = 1 + 2e^t \) is a solution to \( t y' = ty - t \). (Here, \( y \) is the dependent variable representing the function, and \( t \) is the independent variable.) HINT: Write down each side of the equation separately. You must show that the left and right hand sides are equal.

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**Writing Differential Equations**

Differential equations describe relationships involving rates of change. For example, to say that the rate of change of a quantity \( y \) is proportional to the current value of \( y \), we should write the following:

\[
y' = ky
\]

(We can be more explicit if we know more. For example, if \( y \) is increasing at a rate equal to twice the value of \( y \), then we have \( y' = 2y \).)

Example: Write down an equation that says \( y \) *decreasing* at a rate three times the value of \( y \). Assume that our function is positive, so \( y > 0 \).

We can write down many other kinds of relationships:

Example: Let \( y \) be a function that has a constant rate of change 7. Write down a differential equation for this function:

Example: Let \( y \) be a function whose rate of change is proportional to the difference between \( y \) and 5. Write down a differential equation for \( y \):
There are many situations that may be modeled by these ideas.

**Example:** The rate at which a bacteria population grows is proportional to the current number of bacteria. So if we let \( y \) be the number of bacteria at time \( t \), then the differential equation _________ describes the situation.

**Example:** Money in a bank account is continuously compounded at a rate of 1%. (This means that the instantaneous rate of change of the money in the account is equal to 1% of the total value.) Thus if \( A(t) \) is the amount of money in the account at time \( t \), we have a differential equation __________.

**Example:** In a chemical reaction, 13% per day of a substance is used up, and 7 grams per day is added. If \( G(t) \) is the amount of the substance in grams after \( t \) days, then the differential equation which describes the situation is

**Euler’s Method**

We have seen that it is easy to confirm whether or not something is a solution to a differential equation, but how do we find a solution? Except for two special examples we will discuss next time (and a very special type we will discuss later), we will not actually talk about how to solve a differential equation, although much is known about the topic. (The Math Department has many courses devoted to this problem.)

Instead, we will usually approximate solutions to a differential equation using a technique known as Euler’s method. It is a numerical technique, similar to using left or right hand sums to approximate an integral.

We can imagine trying to solve the problem using the Fundamental Theorem. Suppose we know the value of \( y \) at some initial value, such as \( t = t_0 \). Then to find the value of \( y \) at some later point \( t_0 + h \), we could use the following formula:

\[
y(t_0 + h) = y(t_0) + \int_{t_0}^{t_0 + h} y'(t) \, dt.
\]

Since we have some sort of formula for \( y' \), this seems like a great idea! The problem is that our formula for \( y' \) usually involves \( y \) itself, so it is very tricky to find the value of the integral above. About the best we can think of is to use a left-hand sum with a single interval to approximate the integral:

\[
\int_{t_0}^{t_0 + h} y'(t) \, dt \approx y'(t_0) h
\]

Since \( y'(t_0) \) involves only \( y \) at \( t_0 \), we can calculate this. Then we can say that \( t_1 = t_0 + h \), we now know \( y(t_1) \) and can use the same technique to move forward again to estimate \( y(t_1 + h) \), which we can call \( y(t_2) \).
For example, using a step size of $h = 1$, we can estimate that
\[ y(1) \approx y(0) + (1) y'(0), \text{ and} \]
\[ y(2) = y(1) + (1) y'(1). \]
At each point $x$, we can estimate how the function changes over the interval $x$ to $x + h$:
\[ y(x + h) \approx y(x) + h y'(x). \]

Let’s look at an example:

**Example:** Suppose $y(0) = 1$ and $y' = 1/y$. Make a table for the values of $y$ at 0, 1, 2, and 3, if we assume $y'$ is approximately constant over each unit interval.

<table>
<thead>
<tr>
<th>$t$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y(t)$</td>
<td>1</td>
<td>2</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

When we have a differential equation together with an initial condition, like the above with $y' = 1/y$ and $y(0) = 1$, we call this an *initial value problem*.

**Example:** Estimate $y(2)$ if $y(0) = 2$ and $y' = 3y$.

$y(1)$

$y(2)$
Notice that we saw in an earlier example that \( y = 2e^{3t} \) was a solution to \( y' = 3y \), and it is also clear that for this function \( y(0) = 2 \). So \( y = 2e^{3t} \) is the solution we want to this initial value problem. So how good were our estimates?

Well, \( y(1) = 2e^3 \approx 40 \) and \( y(2) = 2e^{3\cdot2} = 2e^6 \approx 807 \).

Our estimates are lousy, and in fact they get worse the further we go! Why is this, and how can we improve our estimates?

Each estimate builds on the one before, so the errors compound. We could improve our estimates by using a smaller step size \( h \) so that each approximation is better. In this problem, this will not help very much.

**Morals:**

1) Euler’s method tends to be less accurate the further you go.
2) Smaller step sizes increase the accuracy.
3) Euler’s does not always approximate a good solution.

Computers of course are very good at doing a large number of calculations, and for that reason, you will be using Excel to implement Euler’s method later. Euler’s method does not *have* to start at 0 of course; we can start adding up changes from any point.

**Example:** Estimate \( y(2) \) if \( y(0) = 3 \) and \( y' = 2y - 4t \).

**Summary**

Today, we have

- Defined a differential equation.
- Verified solutions to differential equations by plugging functions in.
- Estimated solutions to differential equations using Euler’s method to add up all the changes from the derivative.