Lab 1: Simpson's Rule

In this Lab you will investigate how Simpson's rule converges to the value of an integral and begin to develop some ideas about how to determine how many subintervals are needed to make a good approximation.

Recall: Parabolas are symmetric about an axis through the focal point. Simpson's Rule fits a parabola across 2 subintervals.

The numerical integration technique known as "Simpson's Rule" is credited to the mathematician Thomas Simpson (1710-1761) of Leicestershire, England. He also worked in the areas of numerical interpolation and probability theory.

Goals for this lab

- To learn to use some techniques in Excel (including graphing and setting up Simpson's rule) which we will use to analyze data in this class.
- To investigate the convergence of Simpson's rule for different types of functions.
- To develop a worksheet that can be easily used later to approximate various integrals we must deal with.

Important Note: The online help for this lab (see "Labs" on our web site) includes a sample of what your finished worksheet should look like, as well as brief tutorials on setting up Simpson's rule. Make use of as much of this material as you need.

Honor Code Policies

Please read and understand the following before you begin. Honor code policies established by your teacher will govern this lab.

You may complete the lab in a small group of 2-3 people. If you choose to complete the lab in a group, all members must be present (at the same place at the same time), during the completion and write-up of all answers and explanations to all problems in the lab. Each person involved *must* contribute to the work. Each person should in fact spend some time working directly with Excel; it cannot be left to one person to do all of the Excel work in the lab. (In this particular lab, you will complete the set-up for Simpson's Rule and then use it to solve 4 problems. Each person in the group should be at the keyboard typing for at least one of these; other members of the group should still help and be involved even when they are not sitting at the computer.)

Each person who contributed to the lab should get a copy of the completed work to use in answering quiz questions. (We recommend for this reason that you create a complete, neat, and well-organized report, but this is up to you.) No one who did not contribute to the lab should get a copy of this report.

Finally, **each person will take the quiz individually**. When you take the quiz, your work should be your own (based on the lab work you did previously). You are *not* permitted to discuss the questions or answers with anyone other than your instructor under any circumstances. To discuss the questions or answers with anyone else will be considered an Honor Code violation.

Part I: Setting up Simpson's Rule and Checking

In this section, you will set up an Excel worksheet to calculate Simpson's rule for a function using $n = 2, 4, 8, 16$, and 32 subintervals. You will do this in a way that will allow you to easily change the interval and the function later, so you can use the same worksheet for many different problems. (Most of the set up has been done for you.)

Begin by downloading a copy of the Simpson's Rule Calculator worksheet from our web site.

It is under "Lab 1" in "Labs," and the file is called "SimpsonCalc.xlsx". The worksheet has already been set up to apply Simpson's rule with $n = 2, 4, 8,$ and 16 subintervals to the function $f(x) = \sqrt{x}$ on the interval from $a = 1$ to $b = 2$. **The results are approximations to $\int_1^2 \sqrt{x}\, dx$.** Let's take a look at this worksheet:

	A	B	C	D	E	F	G	H	I	J	K
					SimpsonCalc.xls						
1											
2			*x*	*y*	constants		Result				
3	n:	2	1	1	1	1			Interval:		
4	Δx:	0.5	1.5	1.22474487	4	4.89897949			a:	1	
5			2	1.41421356	1	1.41421356	1.218865508		b:	2	
6											
7	n:	4	1	1	1	1					
8	Δx:	0.25	1.25	1.11803399	4	4.47213595			*n*	Δx	Result
9			1.5	1.22474487	2	2.44948974			2	0.5	1.218866
10			1.75	1.32287566	4	5.29150262			4	0.25	1.218945
11			2	1.41421356	1	1.41421356	1.218945157		8	0.125	1.218951
12									16	0.0625	1.218951
13	n:	8	1	1	1	1			32	0.0313	0
14	Δx:	0.125	1.125	1.06066017	4	4.24264069					
15			1.25	1.11803399	2	2.23606798					

This screen shot shows the first two calculations (for $n = 2$ and $n = 4$ subintervals) and the beginning of the third. On the left side of the screen are the calculations. Each calculation corresponds to a table with gray and yellow cells, and one peach cell:

- **Gray cells:** Each grouping of gray cells specifies the number of subintervals (n) and the width of those subintervals (Δx). The numbers n and the widths Δx have already been specified for you in all of the calculations.
- **Yellow cells:** The yellow cells to the right of each set of gray cells show the Simpson's rule calculations for each n. There is a column for x values, a column for y values (function values), and a column for constants. (See "Setting up Simpson's Rule" below for a walk-through of what is in these yellow cells.)
- **Peach cell:** The peach cell at the bottom right of each table contains a SUM command to add up everything in column F, and multiply it by $\Delta x/3$. (Remember, Δx is different for each table, it is given in the gray cell at the left of the table.)

To the right of the calculations are two other tables:

- **White interval box:** There is a white box marked "Interval" that contains cells (J4 and J5) in which you enter the limits a and b of the integral to be estimated. Initially, these are set to 1 and 2, respectively. Try entering different values into these two cells, and note that the calculations throughout the worksheet change automatically to reflect the new limits. (When you're done, reset the values to 1 and 2 for now.)
- **Peach results table:** There is a peach table that collects the results from each of the calculations, including the Δx used. Initially, your file will have a zero in the result for 32 subintervals, because the calculations have not been entered on your worksheet yet for this calculation.

Setting Up Simpson's Rule

Let's take a moment to see how to set up Simpson's rule in Excel. We will look at the table for $n = 4$ on your Simpson's Calculator worksheet. This is initially set up to be an estimate for $\int_1^2 \sqrt{x}\,dx$ with four subintervals.

Recall that using Simpson's rule,

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 4f(x_{n-1}) + f(x_n)\right]$$

where Δx is width of each subinterval and the x_i's are endpoints of n subintervals.

We set up a set of columns. We have a column of x values (cells C7 to C11 in this case) and a column of y values (cells D7 to D11 in table 2). Be sure the **Formula Bar** is checked in the View menu.

Then we have a column of constants. Notice that in Simpson's rule, the middle terms must be multiplied by 4 or 2, alternating. This is why our constants column (column E) in the table for $n = 4$ starts with a 1 (in cell E7), then alternates between 4 and 2 (cells E8 and E10 must be a 4), then ends with another 1 (in cell E11).

Column F contains the product of the function values (column D) times the appropriate constant (from column E).

Finally, the cells in column F (cells F7 to F11 for the second table) are summed in the peach box to the right of the table, and multiplied by $\Delta x/3$ (cell G11 in our example). (Remember that we multiply our sum by $\Delta x/3$ at the end of Simpson's rule.)

Finishing the Tables

All of the entries have been completed for you except for the last ($n = 32$). Here, only the x and Δx entries have been entered.

NOTE: Select cell B42. The formula bar shows that this cell contains: =(J5-J4)/B41, which is $(b − a)/n$. The $ next to the J and the 5 and the J and the 4 keep that cell from changing if you were to drag down the column. When you check the x-values, you can see that in each case we are adding the Δx to the preceding x-value.

You will enter columns D, E, F, and G for this calculation.

Note that in the following steps, you are really just copying the pattern begun in the first three tables. So if you get stuck, examine the entries in these tables.

Follow these steps:

- Fill in the function. To get started, use the function $f(x) = \sqrt{x}$ as was used in the other cases. Excel has a built-in square root function called SQRT(cell number). Enter this function in Column D, using the x-values from Column C. (Examine the entries for $n = 2, 4, 8,$ and 16 if you are not sure how to get started.) Enter function values for the $n = 32$ table.
  We complete the column in one of two ways:
  1. Go back into cell C41 and drag to the bottom of the column to select the entire C column. (In our case, this would be down to cell C73.)
     Under the Edit menu, go to the Fill submenu, and select "Down."

  OR

  2. Go back into cell C41, and get to the bottom right corner, the cursor will become a solid, dark +, now drag down the column. The formula will automatically be copied down.

- Now enter the constants in column E for the table. (The pattern needs to follow the pattern for Simpson's rule. See the examples for $n = 2, 4, 8,$ and 16 if you are stuck.)

- You will now enter a formula to multiply the entries in columns D and E into column F. The formula in F41 is "=D41*E41". Hit return to show the calculation. Complete the column in one of the two ways listed above.

- Finally, sum all the entries in the orange box to the right of the table and multiply by $\Delta x/3$. Your Δx is found in a gray cell to the left of the table. You will use the Excel SUM command. You need to sum up the table entries in column F, multiply by the cell containing Δx, and divide by 3.
  Cell G73 should look like this: =(B42/3)*SUM(F41:F73). See the previous peach cells if you have trouble figuring out what to do.
  Careful! Your Δx will be different depending on what the limits a and b are, so do not type in the value of Δx from the gray cell (it should say ".0313" initially). You must enter the address of the cell for Δx so that the Δx will always be right.

Check your results by comparing your table with the table in the Examples section of the on-line Excel help. (It shows the result of setting up the entire spreadsheet for the function $f(x) = \sqrt{x}$ on the interval $[1, 2]$.)

****** **Note** the way the values from Simpson's rule with different numbers of subintervals seem to **converge to an answer (approach a finite limit)** as the number of subintervals increases. If we look at the Result column in the table of estimations with $n = 2, 4, 8, 16,$ and 32 subintervals, we see that the sequence of answers is converging to a limit. This convergence is fairly quick in this case. In fact, the answer seems to be correct to one decimal place using just two subintervals! ******

Part II: Problems to Work

For each new problem, you should go to Edit in the main menu.

- Choose Move or Copy Sheet . . .
- Be sure to choose Create a Copy
- Then you can choose where you want the new worksheet to be. You may change this one for the new problem and you will still have the original.

You are now ready to do some calculations with this table. Because of the way we have set it up, you can change the entries **a** and **b** in the box at the right, then copy a new formula into the entries in column D, and the worksheet will automatically calculate and tabulate the result of Simpson's rule applied to your integral with $n = 2, 4, 8, 16,$ and 32 subintervals for you to examine.

Problem 1. Try the following steps:

1. Change the entry for **a** to -1, and the entry for **b** to 2. (You will generate a lot of errors in your tables when you do this, since your current function is $f(x) = \sqrt{x}$, and this is undefined for negative x.)

2. Change the formulas (column D) in each table to $3x^2 + 2x + 5$. (You will need to enter this as an Excel formula, so you will need to replace references to x with an appropriate entry in column C.) Make sure you copy this change to every y entry in each of the 5 tables.

You can do this by typing the new formula into the first cell. In cell D3 type: =3*C3^2+2*C3+5. Hit Return.

Then go back to that cell, choose "Copy", then select a range of cells and paste the formula.
OR go back to that cell and get to the bottom right corner where the cursor becomes a solid, dark +, now drag down the column. You will have a number (in this case 5) in each of the values in between but that will not affect the calculations.

3. When you are finished, all five entries in the "Results" column should be the same. (Do you know why? See the on-line help if you're not sure.)

Note: If your entry for $n = 32$ is wrong, but was right in the previous example, you probably typed in a number for Δx instead of typing the address of the cell where Δx appears. Fix this before you do anything else!

Problem 2. We will investigate the convergence of Simpson's rule with different numbers of subintervals for the function $f(x) = 2x^3 + 3x^2 - 4x - 2$, on the interval $[-2, 4]$.

a. Set up your table to perform Simpson's rule with the function $f(x) = 2x^3 + 3x^2 - 4x - 2$ on the interval $[-2, 4]$. In other words, estimate $\int_{-2}^{4} 2x^3 + 3x^2 - 4x - 2 \, dx$ using your Simpson calculation worksheet.

(This involves changing **a** and **b**, and then copying a new formula into the y values of each table.)

b. Examine your results. Larger values of n are supposed to increase the accuracy of Simpson's rule. We already "know" that Simpson's Rule is exact for Quadratic equations (highest power of 2) or less. Do your results surprise you? Hypothesize as to what might have caused the results you see.

Problem 3. Next we will consider $f(x) = 3^x$ on the interval $[-3, 3]$.

a. Enter the function $f(x) = 3^x$ on the interval $[-3, 3]$ into your Simpson's rule worksheet to approximate $\int_{-3}^{3} 3^x \, dx$. (Be sure to change **a** and **b**, and paste the function in all the tables.)

NOTE: To enter an exponential function into Excel: In cell D3 type =3^C3. Then drag down.
Do **NOT** use the EXP command. That is only for e.

b. Examine your results. How quickly do your answers converge to a number?
Do they converge to a number?
When $n = 64$, the result is 24.4276. The results are converging.

c. Make a graph of the function $f(x) = 3^x$ on the interval $[-3, 3]$ using the $n = 32$ table of x and y values.
• Create a graph of $f(x)$ values versus x values
   • Select the appropriate data (all the x and y values, but not the titles of these columns).
   • Choose Charts from the choices listed horizontally directly under the tool box.

- A third menu bar opens. Choose Scatter from the list of choices. (This is the most common graph for this course). Then choose a scatter plot connected with smoothed lines, either with the points showing or not.
- You should click on your chart. With the chart selected:
  - You can move the chart within your excel file.
  - You can add a title and / or label the axes. These are under Chart Layout.
  - Set the title of your chart to "Exponential Function". A legend is not needed in this plot, so you can click on it and delete it.
  - You can move your chart to a new sheet under Move Chart . . . in the main menu. We won't do this.

Notice the y-value increments on this graph. We understand that 2 or 4 parabolas would not be enough to fit this graph well. With $n = 32$ and $n = 64$ our results are finally converging. It takes many parabolas to fit this graph.

Problem 4. Now let's look at the function $f(x) = \pi(0.5 + 0.2\sin(12x))^2$ on the interval $[0.5, 2.5]$.

a. Use the Simpson calculation spreadsheet to approximate the integral $\int_{0.5}^{2.5} \pi(0.5 + 0.2\sin(12x))^2 \, dx$.

NOTE: You can enter π in Excel using **PI()** (with **no** arguments and **no** space between the parentheses). Type exactly as shown. The **sine** function is **SIN(cell number)** in Excel.
Don't forget to use the multiplication symbol ***** everywhere there is multiplication; Excel does not let you leave this out.

b. Graph $f(x) = \pi(0.5 + 0.2\sin(12x))^2$ on $[0.5, 2.5]$. Again, using the $n = 32$ table of x and y values. Title your graph "Sinusoidal Function".

c. Examine your graph carefully. Think about how parabolas would approximate the curve shown. Can you approximate the curve with just a few parabolas, or would it take a lot? Use your results from your table to help make a decision about the number of parabolas needed.

Problem 5. Finally, since you should have some doubt about the accuracy of the last approximation, confirm the quality of your last answer by setting up a table below $n = 32$ using Simpson's rule with 64 subintervals to approximate $f(x) = \pi(0.5 + 0.2\sin(12x))^2$. In this case, you will first need to calculate and enter your own Δx. Then set up an x column, y column, constants column, and a column with a product of y and the constants.
NOTE: You can see how to set up Simpson's rule by examining the tables above, or by looking at the on-line help for this lab.

You now have a Simpson's rule calculator that you can use to approximate other integrals you encounter in this class. Make sure everyone in your group ends up with a copy of the calculator file so they can use it as needed throughout the semester.

Be sure everyone who worked on the lab gets a copy of the completed lab. It is a good idea if you make sure to neatly organize all your graphs, tables, and answers to questions before you print out your results so that you can take the quiz on this lab more easily.