S14 Math 1205 – Test 3 April 30, 2014 NAME:____________________ CRN:______

Use only methods from class. You must show work to receive credit.

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1. (12 pts)

   (a) (10 pts) Starting with $x_0 = 1$, calculate the first two iterations of Newton's method (x_1 and x_2) for $f(x) = -x^2 + 4x - 1$.

   $f'(x) = -2x + 4$

   $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \frac{-(1)^2+4(1)-1}{-2(1)+4} = 0$

   $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 0 - \frac{-(0)^2+f(0)-1}{-2(0)+4} = \frac{1}{4}$

   (b) (2 pts) What is the process in part (a) finding?

   The process in part (a) is finding a root of $f(x)$ via successive approximation.

2. (11 pts) Find the fourth-degree Taylor polynomial, $p_4(x)$, for $f(x) = \cos(x)$ centered at $a = 0$.

$$
\begin{aligned}
p_4(x) &= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \frac{f^{(4)}(a)}{4!}(x-a)^4 \\
&= \cos(0) + \frac{-\sin(0)}{1} \cdot x + \frac{-\cos(0)}{2} \cdot x^2 + \frac{\sin(0)}{6} \cdot x^3 + \frac{\cos(0)}{24} \cdot x^4 \\
&= 1 + 0 + \frac{-1}{2} \cdot x^2 + 0 + \frac{1}{24}x^4 \\
&= 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4
\end{aligned}
$$

3. (9 pts) You are trying to compute the volume of a sphere by measuring its diameter and then using the formula $V = \frac{4}{3}\pi r^3$. The measured diameter is $D = 4$ inches and the error in the measurement process is $dD = 0.02$ inches. Use differentials to estimate the error in computing the volume with this diameter measurement.

$$r = \frac{1}{2}D = 2, \qquad dr = \frac{1}{2}dD = 0.01$$

$$\begin{aligned} dV &= V'(r) \cdot dr \\ dV &= 4\pi r^2 \cdot dr \\ dV &= 4\pi(2)^2 \cdot 0.01 \\ dV &= 0.16\pi. \end{aligned}$$

The error in computing the volume is 0.16π in^3.

4. (16 pts)

(a) (6 pts) State the Mean Value Theorem (MVT) for a function f on a closed interval $[a, b]$.

Suppose that f is

(1) continuous on $[a, b]$ and

(2) differentiable on (a, b).

Then there exists a point c in the interval (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

(b) (10 pts) Let $f(x)$ be a differentiable function. Suppose that there is no point x such that $f'(x) = 1$. Can we have different values a and b such that $f(a) = a$ and $f(b) = b$? Justify your answer using a theorem from class.

Suppose that there are two different values a and b such that $f(a) = a$ and $f(b) = b$. Then, because $f(x)$ is differentiable on $[a, b]$, it is also continuous on $[a, b]$. Therefore, by MVT, there exists a $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a} = \frac{b - a}{b - a} = 1.$$

However, for any x, $f'(x) \neq 1$. In particular, $f'(c) \neq 1$.
Thus, we conclude that we cannot have different values a and b such that $f(a) = a$ and $f(b) = b$.

5. (12 pts) Find the absolute maximum and absolute minimum of $f(x) = xe^{-x}$ on $[-1, 2]$. (Hint: $e \approx 3$)

f is continuous on $[-1, 2]$. Therefore, by the Extreme Value Theorem, f attains an absolute maximum and an absolute minimum on $[-1, 2]$.

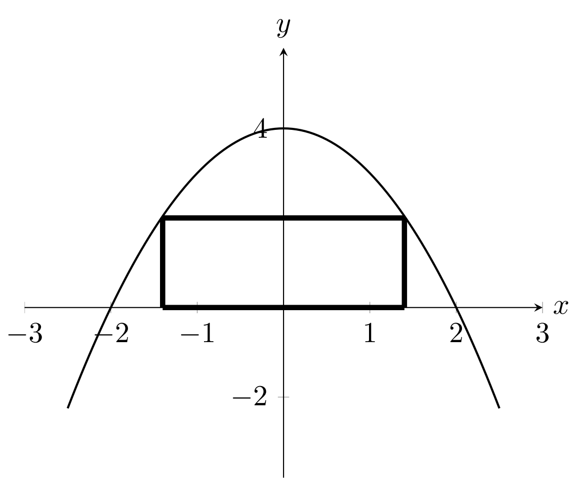
$f'(x) = e^{-x} - xe^{-x} = e^{-x}(1 - x)$

$f'(x) = e^{-x}(1 - x) = 0 \implies x = 1$
$f'(x)$ is defined for all x in $[-1, 2]$.

$f(-1) = -e$
$f(1) = \frac{1}{e}$
$f(2) = \frac{2}{e^2}$

The absolute maximum is $\dfrac{1}{e}$ at $x = 1$.
The absolute minimum is $-e$ at $x = -1$.

6. (13 pts) A rectangle has its base on the x-axis and its upper two vertices on the parabola $y = 4 - x^2$ (see picture). What is the largest area such a rectangle can have?



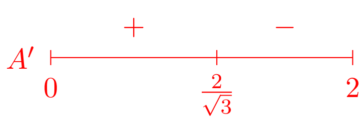
$A = \text{width} \cdot \text{height} = (2x)(y) = (2x)(4 - x^2) = 8x - 2x^3$

The domain of A is $[0, 2]$.

$A' = 8 - 6x^2$

$A' = 0$ when $8 - 6x^2 = 0$, i.e. $x = \pm\dfrac{2}{\sqrt{3}}$. $\therefore x = \dfrac{2}{\sqrt{3}}$ is a critical point ($x = -\dfrac{2}{\sqrt{3}}$ is not in the domain).

A' is always defined on the domain.



Using the number line, we see that A achieves an absolute max at $x = \dfrac{2}{\sqrt{3}}$.

So, the greatest area is $A = \left(2 \cdot \frac{2}{\sqrt{3}}\right)\left(4 - \left(\frac{2}{\sqrt{3}}\right)^2\right) = \frac{32}{3\sqrt{3}}$ units^2.

7. (27 pts) Consider the function $f(x) = \frac{1}{x} + \ln(x)$. (Hint: $\ln(2) \approx 0.7, \ln(3) \approx 1.1, \ln(4) \approx 1.4$)

(a) (2 pts) What is the domain of f?

The domain of $\ln(x)$ is $x > 0$ and the domain of $\frac{1}{x}$ is $x \neq 0$. So the domain of f is $x > 0$.

(b) (4 pts) Find the critical point(s) of f.

$f'(x) = -\frac{1}{x^2} + \frac{1}{x} = \frac{x-1}{x^2}$

$f'(x) = 0$ when $x = 1 \Rightarrow x = 1$ is a critical point

$f'(x)$ is undefined when $x = 0$ which is not in the domain $\Rightarrow x = 0$ is not a critical point

(c) (4 pts) On which interval(s) is f increasing? On which interval(s) is f decreasing?

$x > 1$ or $x \in (1, \infty)$, f is increasing.

$x < 1$ or $x \in (0, 1)$, f is decreasing.

(d) (4 pts) Find the (x, y)-coordinates of all local extrema of f. Indicate if (x, y) is the location of a local minimum or a local maximum.

local minimum: $(1, 1)$.

(e) (5 pts) On which interval(s) is the graph of f concave up? On which interval(s) is it concave down?

$f''(x) = \frac{2}{x^3} - \frac{1}{x^2} = \frac{2-x}{x^3}$.

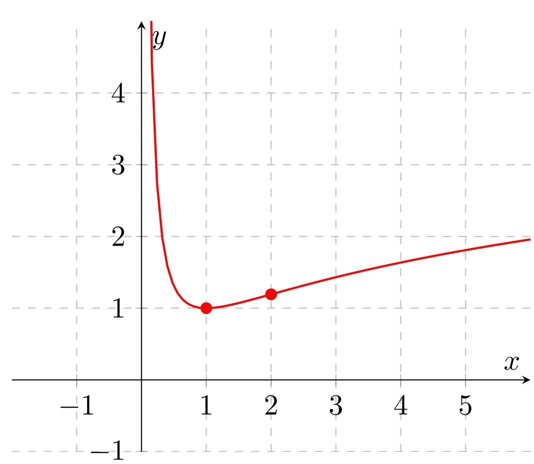
concave up: $(0, 2)$

concave down: $(2, \infty)$

(f) (3 pts) Find the (x, y)-coordinates of any point(s) of inflection of f.

The inflection point of f is $(2, f(2))$ where $f(2) = \frac{1}{2} + \ln 2 \approx 1.2$

(g) (5 pts) Given that $x = 0$ is a vertical asymptote of f, plot the graph of the function f.



Honor Pledge: I have neither given nor received help on this exam. Signed:__________________________