1. (12 pts)

(a) (10 pts) Starting with \( x_0 = 1 \), calculate the first two iterations of Newton’s method (\( x_1 \) and \( x_2 \)) for \( f(x) = -x^2 + 4x - 1 \).

\[
f'(x) = -2x + 4
\]

\[
x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \frac{-1^2 + 4(1) - 1}{-2(1) + 4} = 0
\]

\[
x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 0 - \frac{-0^2 + 4(0) - 1}{-2(0) + 4} = \frac{1}{4}
\]

(b) (2 pts) What is the process in part (a) finding?

The process in part (a) is finding a root of \( f(x) \) via successive approximation.

2. (11 pts) Find the fourth-degree Taylor polynomial, \( p_4(x) \), for \( f(x) = \cos(x) \) centered at \( a = 0 \).

\[
p_4(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \frac{f^{(4)}(a)}{4!}(x-a)^4
\]

\[
= \cos(0) + \frac{-\sin(0)}{1} \cdot x + \frac{-\cos(0)}{2} \cdot x^2 + \frac{\sin(0)}{6} \cdot x^3 + \frac{\cos(0)}{24} \cdot x^4
\]

\[
= 1 + 0 + \frac{-1}{2} \cdot x^2 + 0 + \frac{1}{24} x^4
\]

\[
= 1 - \frac{1}{2} x^2 + \frac{1}{24} x^4
\]
3. (9 pts) You are trying to compute the volume of a sphere by measuring its diameter and then using the formula \( V = \frac{4}{3} \pi r^3 \). The measured diameter is \( D = 4 \) inches and the error in the measurement process is \( dD = 0.02 \) inches. Use differentials to estimate the error in computing the volume with this diameter measurement.

\[
\begin{align*}
   r &= \frac{1}{2} D = 2, \quad dr = \frac{1}{2} dD = 0.01 \\
   dV &= V'(r) \cdot dr \\
   dV &= 4\pi r^2 \cdot dr \\
   dV &= 4\pi (2)^2 \cdot 0.01 \\
   dV &= 0.16\pi.
\end{align*}
\]

The error in computing the volume is 0.16\( \pi \) in\(^3\).

4. (16 pts)

(a) (6 pts) State the Mean Value Theorem (MVT) for a function \( f \) on a closed interval \([a, b]\).

Suppose that \( f \) is

(1) continuous on \([a, b]\) and

(2) differentiable on \((a, b)\).

Then there exists a point \( c \) in the interval \((a, b)\) such that

\[
   f'(c) = \frac{f(b) - f(a)}{b - a}.
\]

(b) (10 pts) Let \( f(x) \) be a differentiable function. Suppose that there is no point \( x \) such that \( f'(x) = 1 \).

Can we have different values \( a \) and \( b \) such that \( f(a) = a \) and \( f(b) = b \)? Justify your answer using a theorem from class.

Suppose that there are two different values \( a \) and \( b \) such that \( f(a) = a \) and \( f(b) = b \). Then, because \( f(x) \) is differentiable on \([a, b]\), it is also continuous on \([a, b]\). Therefore, by MVT, there exists a \( c \in (a, b) \) such that

\[
   f'(c) = \frac{f(b) - f(a)}{b - a} = \frac{b - a}{b - a} = 1.
\]

However, for any \( x \), \( f'(x) \neq 1 \). In particular, \( f'(c) \neq 1 \).

Thus, we conclude that we cannot have different values \( a \) and \( b \) such that \( f(a) = a \) and \( f(b) = b \).
5. (12 pts) Find the absolute maximum and absolute minimum of \( f(x) = xe^{-x} \) on \([-1, 2]\). (Hint: \( e \approx 3 \))

\( f \) is continuous on \([-1, 2]\). Therefore, by the Extreme Value Theorem, \( f \) attains an absolute maximum and an absolute minimum on \([-1, 2]\).

\[
f'(x) = e^{-x} - xe^{-x} = e^{-x}(1 - x)
\]

\[
f'(x) = e^{-x}(1 - x) = 0 \implies x = 1
\]

\( f' \) is defined for all \( x \) in \([-1, 2]\).

\[
f(-1) = -e
\]

\[
f(1) = \frac{1}{e}
\]

\[
f(2) = \frac{2}{e^2}
\]

The absolute maximum is \( \frac{1}{e} \) at \( x = 1 \).

The absolute minimum is \(-e\) at \( x = -1 \).

6. (13 pts) A rectangle has its base on the \( x \)-axis and its upper two vertices on the parabola \( y = 4 - x^2 \) (see picture). What is the largest area such a rectangle can have?

\[
A = \text{width} \cdot \text{height} = (2x)(y) = (2x)(4 - x^2) = 8x - 2x^3
\]

The domain of \( A \) is \([0, 2]\).

\[
A' = 8 - 6x^2
\]

\( A' = 0 \) when \( 8 - 6x^2 = 0 \), i.e. \( x = \pm \frac{2}{\sqrt{3}} \). \( \therefore x = \frac{2}{\sqrt{3}} \) is a critical point (\( x = -\frac{2}{\sqrt{3}} \) is not in the domain).

\( A' \) is always defined on the domain.

\[
\begin{array}{c c c c}
0 & \frac{2}{\sqrt{3}} & 2 \\
A' & + & - & \\
\end{array}
\]

Using the number line, we see that \( A \) achieves an absolute max at \( x = \frac{2}{\sqrt{3}} \).

So, the greatest area is \( A = \left( 2 \cdot \frac{2}{\sqrt{3}} \right) \left( 4 - \left( \frac{2}{\sqrt{3}} \right)^2 \right) = \frac{32}{3\sqrt{3}} \text{ units}^2 \).
7. (27 pts) Consider the function \( f(x) = \frac{1}{x} + \ln(x) \). (Hint: \( \ln(2) \approx 0.7, \ln(3) \approx 1.1, \ln(4) \approx 1.4 \))

(a) (2 pts) What is the domain of \( f \)?
The domain of \( \ln(x) \) is \( x > 0 \) and the domain of \( \frac{1}{x} \) is \( x \neq 0 \). So the domain of \( f \) is \( x > 0 \).

(b) (4 pts) Find the critical point(s) of \( f \).
\[
f'(x) = -\frac{1}{x^2} + \frac{1}{x} = \frac{x - 1}{x^2}
\]
\( f'(x) = 0 \) when \( x = 1 \Rightarrow x = 1 \) is a critical point
\( f'(x) \) is undefined when \( x = 0 \) which is not in the domain \( \Rightarrow x = 0 \) is not a critical point

(c) (4 pts) On which interval(s) is \( f \) increasing? On which interval(s) is \( f \) decreasing?
\( x > 1 \) or \( x \in (1, \infty) \), \( f \) is increasing.
\( x < 1 \) or \( x \in (0, 1) \), \( f \) is decreasing.

(d) (4 pts) Find the \((x,y)\)-coordinates of all local extrema of \( f \). Indicate if \((x,y)\) is the location of a local minimum or a local maximum.
local minimum: \((1, 1)\).

(e) (5 pts) On which interval(s) is the graph of \( f \) concave up? On which interval(s) is it concave down?
\[
f''(x) = \frac{2}{x^3} - \frac{1}{x^2} = \frac{2-x}{x^3}.
\]
concave up: \((0, 2)\)
concave down: \((2, \infty)\)

(f) (3 pts) Find the \((x, y)\)-coordinates of any point(s) of inflection of \( f \).
The inflection point of \( f \) is \((2, f(2))\) where \( f(2) = \frac{1}{2} + \ln 2 \approx 1.2 \)

(g) (5 pts) Given that \( x = 0 \) is a vertical asymptote of \( f \), plot the graph of the function \( f \).

Honor Pledge: I have neither given nor received help on this exam. Signed:___________________________