S14 MATH 1205 – Test 2 2 April 2014 NAME: ____________________ CRN:____________

Use only methods from class. You must show work to receive credit.

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1. (32 pts) Find dy/dx for each of the following:

   (a) (7 pts) $y = \ln(x^3+4) - \tan^{-1}\left(\frac{x}{2}\right)$

   $y' = \frac{1}{x^3+4}\cdot 3x^2 - \frac{1}{(x/2)^2+1}\cdot\frac{1}{2} = \frac{3x^2}{x^3+4} - \frac{2}{x^2+4}$

   (b) (7 pts) $y = \log_2(x)\cdot e^{\cot(7x)}$

   $y' = \frac{1}{\ln(2)x}\cdot e^{\cot(7x)} + \log_2(x)\cdot e^{\cot(7x)}\cdot\big(-\csc^2(7x)\big)\cdot 7$

   (c) (8 pts) $y = \frac{\csc(3x)}{\pi^x}$

   $y' = -3\csc(3x)\cot(3x)\cdot\pi^{-x} + \csc(3x)\cdot(-\ln(\pi)\pi^{-x})$ or $y' = \frac{-3\csc(3x)\cot(3x)\pi^x - \csc(3x)\cdot(\ln(\pi)\pi^x)}{(\pi^x)^2}$

   (d) (10 pts) $y = \big(\sin(x)\big)^x$

$$\begin{aligned}\ln y &= \ln((\sin(x))^x)\\ &= x\cdot\ln(\sin(x))\\ \frac{y'}{y} &= \ln(\sin(x)) + x\cdot\frac{\cos(x)}{\sin(x)}\\ y' &= y\big(\ln(\sin(x)) + x\cdot\cot(x)\big)\\ &= \big(\sin(x)\big)^x\cdot\big(\ln(\sin(x)) + x\cdot\cot(x)\big)\end{aligned}$$

2. (9 pts) Let $f(x) = x^{2013} + e^{2x}$. Find $f^{(2014)}(x)$.

   Taking derivatives, we can see that

$$\begin{aligned}(x^{2013})' &= 2013x^{2012},\\ (x^{2013})'' &= 2013\times 2012x^{2011},\\ &\vdots\\ (x^{2013})^{(2013)} &= 2013\cdot 2012\cdot 2011\cdots 2\cdot 1,\\ (x^{2013})^{(2014)} &= 2013\cdot 2012\cdot 2011\cdots 2\cdot 1\cdot 0 = 0.\end{aligned}$$

$$\begin{aligned}(e^{2x})' &= 2e^{2x},\\ (e^{2x})'' &= 2^2e^{2x},\\ &\vdots\\ (e^{2x})^{(2014)} &= 2^{2014}e^{2x}.\end{aligned}$$

   Hence

$$f^{(2014)}(x) = (x^{2013})^{(2014)} + (e^{2x})^{(2014)} = 2^{2014}e^{2x}.$$

3. (15 pts) For $x > 0$, define $f(x) = x^2e^x$.

   (a) (2 pts) Evaluate $f(x)$ at $x = 1$.

   $f(1) = 1^2\cdot e^1 = 1\cdot e = e.$

   (b) (13 pts) Use a theorem from class to find $(f^{-1})'(e)$. Make sure to state and check the conditions that need to hold in order to use the theorem.

   The domain of $f(x)$ is an interval: $(0,\infty)$. Since $f'(x) = 2xe^x + x^2e^x$, which is positive for $x > 0$, and $f(x)$ is only defined on $(9,\infty)$, $f'(x)$ is always positive (or never zero). So we can use the derivative rule for inverses, which says:

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}.$$

When $x = e$, $f^{-1}(e) = 1$.
Also $f'(x) = 2xe^x + x^2e^x$, hence $f'(1) = 3e$.
Therefore, $(f^{-1})'(e) = \dfrac{1}{f'(1)} = \dfrac{1}{3e}$.

4. (22 pts) The position, in miles, of a car traveling along a horizontal axis is represented by the following function of t, given in hours.

$$s(t) = \frac{t^3}{3} - 3t^2 + 5t + 1, \quad 0 \le t \le 4.$$

Answer each of the following questions with a complete sentence.

(a) (3 pts) Find the displacement of the car from $t = 0$ to $t = 3$.

The displacement is the change of the position during the time, i.e.,

$$\begin{aligned}\Delta s &= s(3) - s(0) = \frac{t^3}{3} - 3t^2 + 5t + 1\Big|_{t=3} - 1\\ &= 9 - 3 \cdot 9 + 15\\ &= -3\text{m},\end{aligned}$$

Therefore, the displacement from $t = 0$ to $t = 3$ is -3 m or 3 m on the left side of the origin.

(b) (3 pts) Find the average velocity of the car from $t = 0$ to $t = 3$.

The average velocity is $v_{\text{avg}} = \dfrac{\Delta s}{\Delta t}$, hence

$$\frac{\Delta s}{\Delta t} = \frac{s(3) - s(0)}{3 - 0} = \frac{-3}{3} = -1.$$

thus the average velocity between $t = 0$ and 3 is -1 m/h or 1 m/h in the negative direction.

(c) (6 pts) At what times between $t = 0$ and $t = 4$ does the car change direction?

The velocity is

$$v(t) = s'(t) = t^2 - 6t + 5,$$

Set $v(t) = 0$, i.e., $(t - 1)(t - 5) = 0$, so $t = 1$ or $t = 5$.
The car stops instantaneously at $t = 1$ and $t = 5$. Only $t = 1$ is in the requested interval.
Since $v(0) > 0$ and $v(2) < 0$, the car changes direction from positive to negative at $t = 1$.

(d) (5 pts) Is the car speeding up or slowing down at time $t = 2$?

The acceleration is $a(t) = v'(t) = 2t - 6 = 2(t - 3)$. When $t = 2$, $a(2) = -2 < 0$. The velocity at this time is $v(2) = 4 - 12 + 5 = -3 < 0$. So the velocity and the acceleration have the same sign. Thus the car is speeding up at this time.

Alternative approach: Near $t = 2$, speed is $|v(t)| = |(t - 1)(t - 5)| = -(t^2 - 6t + 5)$. Thus the derivative of speed is $\frac{d}{dt}|v(t)| = -2t + 6$. So at $t = 2$ $\frac{d}{dt}|v(t)|\big|_{t=2} = -2(2) + 6 = 2 > 0$. Thus the speed is increasing, and the car is speeding up at this time.

(e) (5 pts) Find the total distance traveled by the car from $t = 0$ to $t = 4$.

The conclusion from above tells us that the car changes directions at $t = 2$ and $t = 3$, hence the total distance is calculated based on two sections between $t = 0$ & $t = 1$ and $t = 1$ & $t = 4$:

$$\begin{aligned}s_1 &= s(1) - s(0) = \frac{10}{3} - 1 = \frac{7}{3}\,\text{m},\\ s_2 &= s(4) - s(1) = -\frac{17}{3} - \frac{10}{3} = -9\,\text{m},\end{aligned}$$

hence the total distance is

$$|s_1| + |s_2| = \frac{7}{3} + 9 = \frac{34}{3}\text{m}.$$

5. (13 pts) Define a curve by the equation $x^2 + xy + 5y^2 = 35$. Find the equation of the **normal line** to this curve through the point $(5, 1)$.

Use implicit differentiation, one of two ways:

$$\begin{aligned} 2x + y + xy' + 5 \cdot 2yy' &= 0 \\ xy' + 10yy' &= -(2x + y) \\ y'(x + 10y) &= -(2x + y) \\ y' &= -\frac{2x + y}{x + 10y} \\ \text{At } (5,1) \qquad y' &= -\frac{11}{15} \end{aligned}$$

$$\begin{aligned} 2x + y + xy' + 5 \cdot 2yy' &= 0 \\ \text{At } (5,1) \qquad 10 + 1 + 5y' + 10y' &= 0 \\ 15y' &= -11 \\ y' &= -\frac{11}{15} \end{aligned}$$

The slope of the tangent line is $m_{\text{t}} = y' = -\frac{11}{15}$, so the slope of the normal line is $m_{\text{n}} = -\frac{1}{m_{\text{t}}} = \frac{15}{11}$. The point $(x, y) = (5, 1)$ lies on the normal line, thus

$$\begin{aligned} y - y_0 &= m_{\text{n}}(x - x_0) \\ y - 1 &= \frac{15}{11}(x - 5) \\ y &= \frac{15}{11}(x - 5) + 1 = \frac{15}{11}x - \frac{64}{11} \end{aligned}$$

6. (9 pts) A cylindrical balloon is deflated in such a way that it maintains its shape during deflation. Suppose the radius r is decreasing at a constant rate of 4 cm/s while the height h is decreasing at a constant rate of 2 cm/s. What is the rate of change of the volume when $r = 3$m and $h = 4$m? The volume of a cylinder with radius r and height h is $V = \pi r^2 h$. Answer with a complete sentence.

Working in centimeters, $\frac{dr}{dt} = -4$, $\frac{dh}{dt} = -2$, $r = 300$, $h = 400$

$$\begin{aligned} V &= \pi r^2 h \\ \frac{dV}{dt} &= \pi \left(2r\frac{dr}{dt}h + r^2\frac{dh}{dt} \right) \\ &= \pi \left(2(300)(-4)(400) + (300)^2(-2) \right) \\ &= -1,140,000\pi \end{aligned}$$

The volume of the balloon is decreasing at a rate of $1,140,000\pi$ cm^3/s